Section 1: 3D Lines

In 2D, we commonly expressed straight lines with the formula \( y = mx + b \). We might be tempted to generalize this directly to 3D and write something like \( z = ax + by + c \). We already know, however, that this corresponds to a plane, not a line.

Fortunately, the equation \( y = mx + b \) was not the only representation of a 2D line available to us. Consider the line given by

\[
y = x + 1.
\]

(1)

The position on the line (say, \( \mathbf{r} \)) can be completely described parametrically by three pieces of information:

(a) an initial point on the line (say, \( \mathbf{r}_0 \));

(b) the slope or direction of the line (say, \( \mathbf{v} \)); and

(c) a parameter corresponding to distance (say, \( t \in \mathbb{R} \)).

For example, suppose we are traveling on a perfectly straight road and wish to pass our coordinates on to a relative many states away. It would be enough to tell them where we started (the initial point), which direction we were headed (the slope), and how far we had traveled (the free parameter).

We now use this information to reformulate (1) as a vector equation in parametric form. For notational simplicity, we represent points as vectors. To determine the initial point, we may choose any point on the line. We can choose, for instance, the point vector \( \mathbf{r}_0 = (1, 2) \). To represent the slope \( m = 1 \) as a vector we recall that the slope measures the change in \( y \) per unit change in \( x \). This means we that, for the form \( y = mx + b \), we may always choose the vector \( (1, m) \). For this example, we have \( \mathbf{v} = (1, 1) \). It follows that (1) may be represented in parametric form as

\[
\mathbf{r}(t) = (1, 2) + (1, 1)t = (1 + t, 2 + t).
\]
What should be clear about this example is that it generalizes immediately to higher dimensions in a way the equation $y = mx + b$ does not. We have the following.

**Definition 1**

The **parametric form** of a line $r(t)$ in 3D is given by

$$r(t) = r_0 + tv$$

where $r(t) = \langle x(t), y(t), z(t) \rangle$, $r_0 = \langle x_0, y_0, z_0 \rangle$, and $v = \langle v_1, v_2, v_3 \rangle$. The **symmetric form** of a 3D line is given by:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

**Note:** The symmetric form of the equation can be derived from the parametric form by solving for $t$ in the three coordinates. The symmetric form is generally better for checking whether a specific point lies on a given line.
Example 1

Determine the parametric and symmetric forms of the line through \( P_0 = (2, 3, -1) \) and \( P_1 = (7, 2, 0) \). Does the line travel through \( P_2 = (-8, 5, 3) \) or \( P_3 = (12, 1, 2) \)?

Solution: We may use either \( P_0 \) or \( P_1 \) as our initial point, but we still need to determine \( v \). We have

\[
v = \overrightarrow{P_0 P_1} = P_1 - P_0 = (5, -1, 1).
\]

We may now use the formula to get the parametric form

\[
r(t) = r_0 + tv = (2, 3, -1) + t(5, -1, 1) = (2 + 5t, 3 - t, -1 + t).
\]

It follows that we have \( x = 2 + 5t, \ y = 3 - t, \) and \( z = -1 + t \) for \( t \in \mathbb{R} \). To obtain the symmetric form, we solve for \( t \). We have

\[
t = \frac{x - 2}{5} = \frac{y - 3}{-1} = z + 1.
\]

In order to check whether this line passes through \( P_2 = (-8, 5, -3) \), we need to check the parametric form. We have

\[
\frac{x - 2}{5} = \frac{(-8) - 2}{5} = -2, \quad \frac{y - 3}{-1} = \frac{(5) - 3}{-1} = -2, \quad z + 1 = (-3) + 1 = -2.
\]

Since these equations agree, we have that line does pass through the point \( P_2 = (-8, 5, 3) \) (with \( t = -2 \)). We can also easily check that

\[
(2, 3, -1) + (-2)(5, -1, 1) = (-8, 5, -3).
\]

For \( P_3 = (12, 1, 2) \), we have

\[
\frac{x - 2}{5} = \frac{(12) - 2}{5} = 2, \quad \frac{y - 3}{-1} = \frac{(1) - 3}{-1} = 2, \quad z + 1 = (2) + 1 = 3
\]

Since these do not agree, we may not determine a consistent value of \( t \) for the three equations. It follows that the line does not pass through \( P_3 = (12, 1, 2) \).
Section 2: 3D Planes

Regardless of whether we are in 2D or 3D, lines are fundamentally one-dimensional objections. This can be seen by the parametrization formula (2) which requires one free parameter $t$.

We have already dealt briefly with the most basic two-dimensional object in 3D, that is, a plane. Previously, however, we were only able to handle planes which were defined by coordinate axes (e.g. the $(x, y)$-plane, $(y, z)$-plane, etc.) or those which had one or more variables missing (e.g. $y - z = 2$, etc.). Given our newfound confidence with vector algebra, we will be able to handle far more general cases.

We formally remind ourselves of the definition of a plane.

**Definition 2**

A plane in 3D space is defined by the equation

$$ax + by + cz = d. \tag{3}$$

In order to understand the general plane equation of the form (3), we need the following result.

**Theorem 1**

Consider any two points $P_1$ and $P_2$ lying on a plane with equation (3). Then the vector $\langle a, b, c \rangle$ is orthogonal to $\overrightarrow{P_1P_2}$. That is, we have

$$\overrightarrow{P_1P_2} \cdot \langle a, b, c \rangle = 0.$$ 

**Proof**

This result should seem a little surprising, but it is early in the semester, so the argument cannot be that complicated. In fact, expanding out the given information gives us exactly what we need.

We have that the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ lie on
the plane, so that we have
\[ \begin{align*}
ax_1 + by_1 + cz_1 &= d \\
ax_2 + by_2 + cz_2 &= d.
\end{align*} \]

We can subtract the first expression from the second to obtain
\[ a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0. \]

After a little bit of staring, we can recognize that this is exactly the equation \( \langle a, b, c \rangle \cdot \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = 0. \) We notice that \( \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \) and \( P_1 \) and \( P_2 \) were arbitrary points lying on the plane. It follows that \( \overrightarrow{P_1P_2} \) is orthogonal to \( \langle a, b, c \rangle \), and we are done.

Theorem 1 tells us something very important about planes: the constants \( a, b, \) and \( c \) in the standard form equation (3) determine the normal vector \( \mathbf{n} \) which is orthogonal to the plane! (In fact, normal vectors play a significant role throughout vector calculus, partial differential equations, and the numerous applied fields which use the related tools.)

We have used our understanding of the dot product to understand properties of the standard form (3). It is now the turn of the cross product to determine how to find such a normal vector. We have the following result.
Theorem 2

Suppose we have three points \( P_1 = (x_1, y_1, z_1) \), \( P_2 = (x_2, y_2, z_2) \), and \( P_3 = (x_3, y_3, z_3) \) which lie on a plane, but which do not lie on a common line. Let \( \langle a, b, c \rangle \) denote the cross product of any two of \( \overrightarrow{P_1P_2} \), \( \overrightarrow{P_1P_3} \) and \( \overrightarrow{P_2P_3} \). Then the equation of the plane is given by

\[
ax + by + cz = d
\]  

(4)

where \( d := ax_i + by_i + cz_i \) and \( i = 1, 2, 3 \) may be chosen arbitrarily.

Proof. We omit the full proof, which requires some elements from linear algebra. The general idea is, given any two vectors in 3D, we know that the cross product is orthogonal to both vectors by our cross product properties. It follows that we can determine \( \langle a, b, c \rangle \) by taking the cross product of any two of \( \overrightarrow{P_1P_2} \), \( \overrightarrow{P_1P_3} \) or \( \overrightarrow{P_2P_3} \). If we then take any two points \( P = (x, y, z) \) and \( P_i = (x_i, y_i, z_i), i = 1, 2, 3 \), where \( P \) lies on the plane, we arrive at

\[
\langle a, b, c \rangle \cdot \overrightarrow{P_iP} = a(x - x_i) + b(y - y_i) + c(z - z_i) = 0.
\]

The required form (4) follows immediately.

Note: We now have all the information required to obtain the form (3)! We need to execute the following steps:

1. Determine the cross product of any two of \( \overrightarrow{P_1P_2} \), \( \overrightarrow{P_1P_3} \) or \( \overrightarrow{P_2P_3} \) to determine the values \( a \), \( b \), and \( c \).
2. Rearrange the form (4) where \( P_i \) may be chosen arbitrarily to obtain the value for \( d \).

Example 2

Determine the equation of the plane in the form (3) for the plane through the three points \( P_1 = (1, 0, 3) \), \( P_2 = (2, 1, -1) \) and \( P_3 = (-1, 3, 5) \).

Solution: We need to construct two vectors lying on the plane and
may do so from any combination of the three given points. We pick

\[ \mathbf{v}_1 = \overrightarrow{P_1P_2} = (2, 1, -1) - (1, 0, 3) = (1, 1, -4) \]
\[ \mathbf{v}_2 = \overrightarrow{P_1P_3} = (-1, 3, 5) - (1, 0, 3) = (-2, 3, 2). \]

To determine the values of \( a, b, \) and \( c, \) we need to compute the cross product of these two vectors. We have

\[
\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 1 & -4 \\
-2 & 3 & 2
\end{vmatrix} = \begin{vmatrix}
1 & -4 \\
3 & 2
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
1 & -4 \\
-2 & 2
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
1 & 1 \\
-2 & 3
\end{vmatrix} \mathbf{k} = 14 \mathbf{i} + 6 \mathbf{j} + 5 \mathbf{k} = (14, 6, 5).
\]

It follows that we have \( a = 14, \) \( b = 6, \) and \( c = 5. \) Selecting \( P_1 = (1, 0, 3), \) the final equation is given by

\[ 14(x - 1) + 6(y - 0) + 5(z - 3) = 0 = 14x + 6y + 5z = 29 \]

so that \( d = 29. \)

It can be easily checked that the three points \( P_1 = (1, 0, 3), \) \( P_2 = (2, 1, -1) \) and \( P_3 = (-1, 3, 5) \) satisfy this equation. As additional verification of the method, consider doing the problem with the vector \( \overrightarrow{P_2P_3} \) instead of \( \overrightarrow{P_1P_2} \) or \( \overrightarrow{P_1P_3}. \)

**Example 3**

Determine the parametric form of the line given by the intersection of the following planes:

Plane 1: \( x + 2y - 5z = 2 \)
Plane 2: \( -2x + y - z = 0. \)

**Solution:** This is certainly a different type of problem, but we already have all of the tools required to solve it. The first thing we need to recognize is that, by definition, the line of intersection lies on both of
the planes, and in particular, the property of vectors being orthogonal to the vectors \( \langle a, b, c \rangle \). For this example, we have that, for any points \( P_1 \) and \( P_2 \) lying on the line of intersection, we have \( \overrightarrow{P_1 P_2} \) is orthogonal to \( \langle 1, 2, -5 \rangle \) and \( \langle -2, 1, -1 \rangle \).

In order to determine this vector, we compute

\[
\mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2 & -5 \\
-2 & 1 & -1
\end{vmatrix}
= 2\mathbf{j} - 5\mathbf{i} + 1\mathbf{k} + 1\mathbf{j} - 2\mathbf{i} + 2\mathbf{k}
= 3\mathbf{i} + 11\mathbf{j} + 5\mathbf{k} = \langle 3, 11, 5 \rangle.
\]

It follows that \( \langle 3, 11, 5 \rangle \) is parallel to the line we want. In order to determine the parametric form, we simple need to find a point on the line—that is, a point common to both planes. There are clearly a great many points to choose. We can narrow the possibilities by selecting \( z = 0 \). This gives

\[
\begin{align*}
x + 2y &= 2 \\
-2x + y &= 0.
\end{align*}
\]

This has the solution \( x = 2/5 \) and \( y = 4/5 \) so that the \( \mathbf{r}_0 = (2/5, 4/5, 0) \).

It follows that

\[
\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 2/5 + 3t, 4/5 + 11t, 5t \rangle.
\]

A visual representation can be given by Maple:
Suggested Problems

1. Determine the parametric and symmetric forms of the lines through the following pairs of points:

   (a) $P_1 = (1, 0)$ and $P_2 = (0, 1)$
   (b) $P_1 = (1, 0, 0)$ and $P_2 = (0, 1, -1)$
   (c) $P_1 = (2, -1, 8)$ and $P_2 = (0, 1, 3)$
   (d) $P_1 = (1, 1, -2)$ and $P_2 = (5, -3, 1)$

2. If possible, determine the equation of the plane through the following triple of points. If this is not possible, explain why.

   (a) $P_1 = (0, 0, 0)$, $P_2 = (1, 0, 0)$, and $P_3 = (0, 1, 0)$
   (b) $P_1 = (1, 3, -1)$, $P_2 = (2, -1, 0)$, and $P_3 = (0, 0, 2)$
   (c) $P_1 = (1, 2, 0)$, $P_2 = (0, 1, 2)$, and $P_3 = (3, 4, -4)$

3. Determine the parametric form of the line of intersection formed by the following two planes. (If the intersection is not a line, explain what it is.)

   (a) \[ \begin{align*}
   \text{Plane 1:} & \quad 3x - y - z = 1 \\
   \text{Plane 2:} & \quad x + z = 0
   \end{align*} \]
   (b) \[ \begin{align*}
   \text{Plane 1:} & \quad x + y - 2z = 2 \\
   \text{Plane 2:} & \quad -2x - 2y + 4z = -4
   \end{align*} \]
   (c) \[ \begin{align*}
   \text{Plane 1:} & \quad x - y - z = 0 \\
   \text{Plane 2:} & \quad -x - y + z = 0
   \end{align*} \]
   (d) \[ \begin{align*}
   \text{Plane 1:} & \quad 2x - y + z = 0 \\
   \text{Plane 2:} & \quad 2x - y + z = 1
   \end{align*} \]