Section 1: Inflow / Outflow Models

One application of first-order linear differential equations is modeling the amount (or concentration) of a substance in a well-stirred tank/vessel subject to constant in-flow and out-flow. Common simple applications are:

- an industrial mixing tank with an entry pipe (pumping the chemical of interest in) and an exit pipe;
- a lake with an inflow (say, a river) feeding a pollutant from upstream and an outflow (also, a river) flowing downstream;
- a tub or sink with a steady inflow (say, a faucet) and a steady outflow (say, a drain).

In all cases, we are interested in modeling the amount (or concentration) of substance \( x \) in the mixing vessel over time.

To model this problem, we first translate the description into mathematical equations. The fundamental equation we will use to model the change of substance in the mixing vessel will be

\[
\text{[rate of change]} = \text{[rate in]} - \text{[rate out]}.
\]

That is to say, at each instance in time, we believe that the rate of change of the overall amount of the quantity of interest to equal the amount that is flowing in minus the amount that is flowing out.
To characterize the inflow rate, we need to know the overall flow rate in and the concentration of the quantity of interest within that in-flowing mixture. This could be the amount of pollutant in an inflowing stream, or the amount of chemical diluted in an inflowing pipe. The rate of substance flowing in is given by

\[
\text{[rate in]} = \text{[volume rate in]} \times \text{[concentration]}
\]

since

\[
\text{[volume rate in]} \times \text{[concentration]} = \left(\frac{\text{volume}}{\text{time}}\right) \times \left(\frac{\text{amount}}{\text{volume}}\right) = \left(\frac{\text{amount}}{\text{time}}\right).
\]

The outflow is slightly different. We will assume for simplicity that the vessel is well-mixed so that that the concentration of the quantity of interest is the same everywhere. It is reasonable to assume that, regardless of where the outflow is located and no matter how quickly it is flowing, we have

\[
\text{[rate out]} = \text{[concentration]} \times \text{[volume rate out]} = \left(\frac{\text{amount}}{\text{volume}}\right) \times \left(\frac{\text{volume}}{\text{time}}\right)
\]

since

\[
\left(\frac{\text{amount}}{\text{volume}}\right) \times \text{[volume rate out]} = \left(\frac{\text{amount}}{\text{volume}}\right) \times \left(\frac{\text{volume}}{\text{time}}\right) = \left(\frac{\text{amount}}{\text{time}}\right).
\]

Letting \(A\) denote the amount of substance in the tank and \(V(t)\) denote the current volume, we have the combined model

\[
\frac{dA}{dt} = \text{[volume rate in]} \times \text{[concentration in]} - \frac{A}{V(t)} \times \text{[volume rate out]}.
\]

**Note:** The key difference between the inflow and outflow rates is that the *amount* and *volume* in the outflow rate depend upon the *current* amount and volume in the mixing vessel. In the inflow tank, these quantities are either controlled (for mixing tanks) or known (for rivers and streams). Notice also that if the volume of the in-flow and the volume of the out-flow do not balance, the volume of the tank may be a dynamic function of time (imagine filling a bathtub, or emptying a mixing tank).
Example 1

Suppose that there is a factory built upstream of a lake with a volume of 0.5 km$^3$. The factory introduces a new pollutant to a stream which pumps 1 km$^3$ of water into the lake every year. Suppose that the net outflow from the lake is also 1 km$^3$ per year and that the concentration of the pollutant in the inflow stream is 200 kg/km$^3$.

(a) Set up an initial value problem for the amount of pollutant in the lake and solve it.

(b) Assuming there is initially no pollutant in the lake, how much pollutant is there at one month?

(c) What is the limiting pollutant level?

Solution: We need to set up the model in the form \([\text{rate of change}] = [\text{rate in}] - [\text{rate out}].\) If we let \(A\) denote the amount of the pollutant (in kg), we have

\[
[\text{rate of change}] = \frac{dA}{dt}.
\]

In order to determine the rate in, we notice that the amount (in kg) coming from the inflow can be given by

\[
[\text{rate in}] = [\text{volume rate in}] \times [\text{concentration in}]
= (1 \text{ km}^3/\text{year})(200 \text{ kg/km}^3) = 200 \text{ kg/year}.
\]

The rate out is given by

\[
[\text{rate out}] = [\text{volume rate out}] \times [\text{concentration out}]
= (1 \text{ km}^3/\text{year}) \left( \frac{A}{0.5} \text{ kg/km}^3 \right) = 2A \text{ kg/year}.
\]

We can see the units have worked as desired. We can drop them and just focus on the initial value problem

\[
\frac{dA}{dt} = 200 - 2A, \quad A(0) = A_0.
\]
This is a first-order linear differential equation which in standard form is given by
\[ \frac{dA}{dt} + 2A = 200. \]

We can see that we have \( p(x) = 2 \) and \( q(x) = 200 \). The necessary integration factor is
\[ \rho(t) = e^{\int 2 \, dt} = e^{2t} \]
so that we have
\[ e^{2t} \frac{dA}{dt} + 2e^{2t}A = 200e^{2t} \]
\[ \Rightarrow \frac{d}{dt} [e^{2t}A] = 200e^{2t} \]
\[ \Rightarrow e^{2t}A = \int 200e^{-2t} \, dt = 100e^{2t} + C \]
\[ \Rightarrow A(t) = 100 + Ce^{-2t}. \]

In order to solve for \( C \), we use \( A(0) = A_0 \) to get
\[ A(0) = A_0 = 100 + C \quad \Rightarrow \quad C = A_0 - 100. \]

This gives the solution
\[ A(t) = 100 + (A_0 - 100)e^{-2t}. \]

For this form, we can easily answer part (b). Given an initial pollutant level of zero (i.e. \( A_0 = 0 \)), we have
\[ A(t) = 100 - 100e^{-2t}. \]

After one month has passed, we have \( t = 1/12 \) so that the amount of pollutant is given by
\[ A(1/12) = 100 - 100e^{-2(1/12)} \approx 15.3528 \text{ kg}. \]

We can also easily determine the limiting pollutant level by evaluating
\[ \lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left[ 100 + (A_0 - 100)e^{-2t} \right] = 100. \]
In other words, no matter what the initial amount is in the lake, we will always converge toward 100 kg of pollutant distributed throughout the lake. This makes sense. The limiting level is going to be when the rate in and the rate out are balanced. That occurs for this model when $200 = 2A$ which implies $A = 100$.

**Example 2**

Consider a 50 gallon tank which is initial filled with 20 gallons of brine (salt/water mixture) with a concentration of $1/4$ lbs/gallon of salt. Suppose that there is an inflow tube which infuses 3 gallons of brine into the tank per minute with a concentration of 1 lbs/gallon. Suppose that there is an outflow tube which flows at a rate of 2 gallons per minute.

(a) Set up and solve a differential equation for the amount of salt in the tank.

(b) How much salt is in the tank when the tank is full?

**Solution:** This is slightly different than the previous example because the volume of mixture in the tank changes because the inflow and outflow volume rates are different. There is more mixture flowing into the tank than flowing out. Nevertheless, we can incorporate this into our model by noting that the volume of the tank at time $t$ can be given by

$$V(t) = 20 + (3 - 2)t = 20 + t.$$  

We can now complete the model as before. We have

$$\frac{dA}{dt} = (3)(1) - (2)\frac{A}{20 + t} = 3 - \frac{2A}{20 + t}, \quad A(0) = 20(1/4) = 5.$$  

Again, this is a first-order linear differential equation. We can solve it by rewriting

$$\frac{dA}{dt} + \left(\frac{2}{20 + t}\right)A = 3$$

and determining the integrating factor

$$\rho(t) = e^{\int \frac{2}{20 + t} \, dt} = e^{\ln(20 + t)} = (20 + t)^2.$$
This gives

\[(20 + t)^2 \frac{dA}{dt} + 2(20 + t)A = 3(20 + t)^2\]

\[\implies \frac{d}{dt}[(20 + t)^2 A] = 3(20 + t)^2\]

\[\implies (20 + t)^2 A = (20 + t)^3 + C\]

\[\implies A(t) = (20 + t) + \frac{C}{(20 + t)^2}.\]

Using the initial condition \(A(0) = 5\), we have

\[A(0) = 5 = 20 + \frac{C}{400} \implies C = -6000\]

so that the particular solution is

\[A(t) = (20 + t) - \frac{6000}{(20 + t)^2}.\]

To answer the question of how much salt will be in the tank when the tank is full, we notice that the tank will be full when \(V(t) = 20 + t = 50\), which implies \(t = 30\) (i.e. it will take thirty minutes). This gives

\[A(30) = (20 + 30) - \frac{6000}{(20 + 30)^2} = 50 - \frac{6000}{2500} = 47.6.\]

It follows that there will be 47.6 lbs of salt in the tank when it is full.

**Suggested Problems**

1. Consider a mixing tank with a total volume of 20 gallons, initially filled with 10 gallons of pure water. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water) mixture at a rate of 4 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallon per minute.

   (a) Use the given information to derive a differential equation which
models the amount of salt in the tank.

(b) Find the general solution of the differential equation derived in part (a).

(c) How much salt is in the tank when it is full?

2. Consider a filled mixing tank with a volume of 20 gallons. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water mixture) at a rate of 2 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallons per minute.

(a) Use the given information to derive a differential equation which models the amount of salt in the tank.

(b) Find the general solution of the differential equation derived in part (a).

(c) Suppose there is initially no salt in the tank. How much salt is in the tank after ten minutes?

3. Suppose a factory is built upstream of a lake with a volume of 0.5 km$^3$. The factory introduces a pollutant into the upstream water system. Suppose the affected water system pumps 0.25 km$^3$ of water into the lake each year and the downstream water system removes water from the lake at the same rate. Suppose the concentration of pollutant in the feeding water system is 40 kg/km$^3$.

(a) Set up a first-order differential equation which models the amount of pollutant (in kg) in the lake.

(b) Suppose that there is initially no pollutant in the lake. How much pollutant is in the lake after (i) one month; (ii) seven months; (iii) five years? What is the limiting amount of pollutant in the lake?

(c) Suppose now that the inflow and outflow rates of the upstream and downstream water systems vary based on the seasons. Suppose this variance can be modeled by the form $[\text{volume rate in}] = [\text{volume rate out}] = 0.25 + 0.25 \cos(2\pi t)$. Derive the corresponding first-order differential equation which models the amount of pollutant (in kg) in Lake Mendota.

(d) Suppose that there is initially no pollutant in the lake. Under the assumptions of part (c), determine how much pollutant is in the lake after (i) one month; (ii) seven months; (iii) five years. What
is the limiting amount of pollutant in the lake? Is it the same as in part (b)? Does it converge to this value faster or slower than part (b)? Provide a brief explanation for the observed differences.