Math 133A, September 22:
Review of Linear Algebra

Section 1: Linear Algebra

We will soon begin dealing with interdependent systems of first-order differential equations. Before we can begin such consideration, however, it is important to consider the study of linear systems of equations in general, which is the study of linear algebra. In particular, we will need to define a few new objects, called matrices and vectors, and describe how they interact. Buried inside these objects will be the individual quantities (e.g. variables, functions, derivatives, etc.) which we have been interested in up to this point in the course.

The advantage of briefly introducing linear algebra now is two-fold:

1. It will allow us to condense expressions in a way which will make a first-order system of differential equations analogous to the first-order differential equations studied to date.

2. Theoretical results from linear algebra will be essential in deriving solutions to all but the simplest of such systems.

Of course, linear algebra is not a primary topic of Math 133A. We will introduce only the topics which are necessary for our study, and for theoretical and computational simplicity, will limit ourselves to two-dimensional systems. (The interested student is encouraged to take Math 129A for a fuller introduction to the area.)

Section 2: Matrices & Vectors

The basic objects of study in linear algebra are matrices and vectors, which we will (usually) consist of elements drawn from the real numbers \( \mathbb{R} \).
Definition 1

A matrix $A \in \mathbb{R}^{m \times n}$ is defined to be

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

where $a_{ij} \in \mathbb{R}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. A vector $v \in \mathbb{R}^m$ is defined to be

$$v = (v_1, v_2, \ldots, v_m)$$

where $v_i \in \mathbb{R}$ for all $i = 1, \ldots, m$.

It is important to note that, for each matrix entry $a_{ij}$, the first index $i$ corresponds to the row while the second index $j$ corresponds to the column. This is standard across all disciplines which use linear algebra (which is a lot!). Vectors will alternatively be denoted by bold-faced (e.g. $\mathbf{v}$) or with an arrow overtop (e.g. $\vec{v}$). For graphical clarity, bold-faced will be favored in the online notes while the arrow notation will be favored in class.

Example 1

A matrix can be thought of as a rectangular grid of numbers. For example,

$$A = \begin{bmatrix}
    2 & 0 & -1 & 0 \\
    1 & \frac{1}{2} & 6 & -2
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}$$

are matrices. A vector can be thought of as a matrix with either one row (a row vector) or one column (a column vector). For example, if we have

$$\mathbf{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

then $\mathbf{v}$ is a row vector, and $\mathbf{w}$ is a column vector.

Since matrices consist of entries which are real numbers, which have
well-defined operations $+,-,\times,\div$, it is natural to ask whether matrices have similar operations. We define the following.

**Definition 2**
Consider two matrices $A, B \in \mathbb{R}^{m \times n}$, and a constant $c \in \mathbb{R}$. Then the matrix $A + B \in \mathbb{R}^{m \times n}$ is the matrix with entries

$$[A + B]_{ij} = a_{ij} + b_{ij}$$

and the matrix $cA \in \mathbb{R}^{m \times n}$ is the matrix with entries

$$[cA]_{ij} = ca_{ij}$$

for all $i = 1, \ldots, m$, and $j = 1, \ldots, n$.

**Example 2**
Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}. $$

By the definition of matrix addition, we have

$$A + 2B = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2(0) & 2 + 2(-1) \\ -3 + 2(1) & 1 + 2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}. $$

**Note:** It is important to recognize that the operation $A + B$ is only defined for matrices $A$ and $B$ which have the same dimensions. We may not, for instance, add two matrices with the dimensions $2 \times 3$ and $2 \times 5$. In such a case, we say the operation is not defined.
Section 3: Matrix Multiplication

We will also need to define multiplication for matrices. This is the first operation which is not as intuitive as applying the standard operation component-wise to the relevant matrices. We have the following definition.

**Definition 3**
Suppose $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$. Then the matrix $AB \in \mathbb{R}^{m \times n}$ is the matrix with entries

$$[AB]_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$ 

Alternatively, we may think of matrix multiplication as consisting of multiplying rows of the first matrix component-wise with columns of the second, and then adding the results. The result of this operation is then placed in the row and column of the new matrix corresponding to the row and column which were just multiplied together.

**Example 3**
For the $A$ and $B$ defined in Example 2, we have

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(0) + (2)(1) & (1)(-1) + (2)(2) \\ (-3)(0) + (1)(1) & (-3)(-1) + (1)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}.$$ 

**Note:** The operation of matrix multiplication is only defined if the first matrix has *exactly* the same number of columns as the second matrix has rows. Otherwise, the matrix product $AB$ is not defined. It can also be checked that, it is not generally the case that $AB = BA$ (check with the previous example!). In mathematical terminology, we say that
matrix multiply is **noncommutative**. In plain English, we say **order matters**. This is a significant distinction with multiplication of real numbers, where $xy = yx$ holds trivially.

### Section 4: Inverses and Determinants

A particularly important matrix is the *identity matrix*, which for $2 \times 2$ systems is defined as $I \in \mathbb{R}^{2 \times 2}$ where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

This matrix has the property that $AI = A$ and $IA = A$ for any matrix for which the multiplication is defined. Consequently, in the theory of linear algebra, the identity matrix can be thought of as being analogous to the value *one* in standard algebra.

An important question is whether, given a square matrix $A$, we can find a matrix which multiplies to give the identity matrix. That is to say, can we find a matrix $B$ such that

$$AB = I.$$ 

The most important application of such a matrix is it allows an analogue to *division* for real numbers to be defined for matrices.

**Definition 4**

Consider a matrix $A \in \mathbb{R}^{2 \times 2}$ defined by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

Then the **inverse** matrix $A^{-1} \in \mathbb{R}^{2 \times 2}$ is defined by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ 

and has the property that $A^{-1}A = AA^{-1} = I.$
Proof

The identities can be checked directly. We have that
\[
A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

\[
= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

The other direction can be checked similarly.

The 2 \times 2 inverse matrix \(A^{-1}\) has many nice properties. Most notably, it is guaranteed to be an inverse on the right and on the left (\(A^{-1}A = I\) implies \(AA^{-1} = I\), and vice versa), and it is unique (i.e. given a matrix \(A\) there exactly one matrix satisfying \(A^{-1}A = AA^{-1} = I\)).

An immediate consequence of this formula is that not every matrix has an inverse (i.e. not all matrices are invertible). This follows from the fact that \(ad-bc\) may be zero, in which case the formula tells us to divide by zero. (This could have been anticipated by our interpretation of inverses as the matrix analogue of division!) The quantity telling us to avoid dividing by zero is important enough in the theory of linear algebra that it is given its own name.

Definition 5

The **determinant** of a matrix \(A \in \mathbb{R}^{2 \times 2}\) is defined by
\[
\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc.
\]

Many properties follow from the value of the determinant, but the one which is more readily apparent (and frequently used) is that a matrix is invertible if and only if \(\det(A) \neq 0\). This is true not just for 2 \times 2 matrices, but for any square matrix of any size. We note, however, that the form for the determinant is more complicated in higher dimensions.
Example 4

Determine whether the following matrices are invertible and, if so, find the inverse:

\[ A = \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}. \]

**Solution:** We can simply apply the formula. We first need to compute the determinants. We have

\[ \det(A) = 2(-3) - (-5)(1) = -6 + 5 = -1 \]

and

\[ \det(B) = 2(2) - (-1)(-4) = 4 - 4 = 0. \]

It follows that \( A \) is invertible and \( B \) is not. The inverse of \( A \), \( A^{-1} \), can be computed by the formula as

\[ A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & -2 \end{bmatrix}. \]

We can easily verify this is correct by computing

\[ A^{-1}A = \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \]

**Note:** To see how inverses are related to division, consider being asked to solve the algebraic expression \( ax = b \) for \( x \). We immediately recognize that we need to divide both sides by \( a \). This is the same as multiplying by the reciprocal of \( a \), which is \((1/a)\). Evaluating step-by-step, we have

\[ ax = b \implies \left( \frac{1}{a} \right) ax = \left( \frac{1}{a} \right) b \implies 1 \cdot x = \frac{b}{a} \implies x = \frac{b}{a}. \]

Now consider the analogous matrix expression \( Ax = b \). In order to solve for the vector \( x \), we cannot simply divide by \( A \) because matrix division is not formally defined. We can, however, multiply by the
inverse $A^{-1}$. We have

$$Ax = b \implies A^{-1}Ax = A^{-1}b \implies Ix = A^{-1}b \implies x = A^{-1}b.$$