Instructions

1. Fill out this cover page completely and affix it to the front of your submitted assignment.

2. Staple your assignment together and answer the questions in the order they appear on the assignment sheet.

3. You are encouraged to collaborate on assignment problems but you must write up your assignment independently. Copying is strictly forbidden!

<table>
<thead>
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<th>Correctness</th>
<th>/20</th>
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Grader Initials:
Separable and First-Order Linear DEs

Q1: Determine the general solution of the following differential equations.
[Note: If $y$ cannot be solved for, leave the solution in implicit form.]

(a) $y' = \frac{y^2}{1 + x^2}$
(b) $y' = \frac{x^2}{1 + y^2}$
(c) $y' = \frac{x}{y} + xy$
(d) $y' = \frac{y^2 - 3y + 2}{x}$

Q2: Solve the following initial value problems.

(a) \[
\begin{align*}
y' &= \frac{y + x}{1 + x} \\
y(0) &= 1
\end{align*}
\]
(b) \[
\begin{align*}
y' &= -2\frac{y}{x} + \ln(x) \\
y(1) &= 0
\end{align*}
\]

Q3: Consider $y' = \sqrt{1 - y^2}$.

(a) Sketch the slope field.
(b) Find a solution $y(x)$ of the differential equation over the domain $C - \pi/2 < x < C + \pi/2$ where $C$ is an arbitrary (integration) constant. Superimpose this solution on the slope field found in part (a).
(c) Notice that $y(x) = -1$ is a solution in the range $x \leq C - \pi/2$ and $y(x) = 1$ is a solution in the range $x \geq C + \pi/2$. Comment on the uniqueness of solutions over the entire domain. [Hint: Consider initial conditions $y(x_0) = 1$ and $y(x_0) = -1$ for any $x_0 \in \mathbb{R}$.]

Q4: Consider $y' + y = 2 \cos(x)$.

(a) Sketch the slope field.
(b) Based on an inspection of the slope field, describe how solutions behave for large $x$.
(c) Find the general solution $y(x)$. Compare the behavior as $x \to \infty$ to that predicted in part (b). [Hint: If it helps, $\sin(x) + \cos(x) = \sqrt{2} \cos(x - \pi/4)$]

Q5: According to Newton’s law of cooling/heating, the rate of a body’s temperature change is proportional to the difference between the body’s temperature and the forcing temperature. Suppose we apply a heat to the bottom of a tank of liquid which is divided into two chambers:
Suppose the burner forces the temperature in the first chamber, and the heat in the first chamber forces the temperature in the second chamber. Assuming the temperature equilibrates instantaneously in the chambers, we arrive at the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= k(T - x) \\
\frac{dy}{dt} &= k(x - y)
\end{align*}
\]

where \( k > 0 \) is (roughly) the heat capacity of the liquid, and \( T \) is the temperature of the burner.

(a) For the parameter values \( k = 1 \) and \( T = 400^\circ F \), solve for the temperature profiles, \( x(t) \) (first chamber) and \( y(t) \) (second chamber). Suppose that the chambers are initially at \( 100^\circ F \) (i.e. \( x(0) = y(0) = 100 \)). \([Hint: \text{ Solve for } x(t) \text{ and then use this solution to solve for } y(t)!]\)

(b) What is the long-term behavior of the temperature in the two chambers? Does this make sense in terms of the physical system described?

(BONUS) Assume that the burner is not a constant temperature but rather oscillates according to \( T = 400 + 100 \sin(t) \). Determine the solutions \( x(t) \) and \( y(t) \) using the other values from part (a). What is the long-term behavior in the two chambers?