MATH 133A, Fall 2015, Assignment 5
Due date: Thursday, October 8 (in class)

Name (printed):  
SJSU Student ID Number:  

Instructions

1. Fill out this cover page completely and affix it to the front of your submitted assignment.

| Correctness | /15 |

2. Staple your assignment together and answer the questions in the order they appear on the assignment sheet.

| Completeness | /5 |

3. You are encouraged to collaborate on assignment problems but you must write up your assignment independently. Copying is strictly forbidden!

| Total: | /20 |
| Bonus: | /3 |

Grader Initials:  

Q1: Consider the matrices
\[ A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}. \]
Verify that (a) \((AB)C = A(BC)\) and (c) \(A(B + C) = AB + AC\).

Q2: Determine the eigenvalues, eigenvectors, and (if necessary) generalized eigenvectors of the following matrices.

(a) \[ \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} \]
(b) \[ \begin{bmatrix} -5 & -1 \\ 4 & -1 \end{bmatrix} \]
(c) \[ \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \]
(d) \[ \begin{bmatrix} 5 & -9 \\ 10 & -1 \end{bmatrix} \]

Q3: Consider the matrix
\[ A = \begin{bmatrix} -7 & -4 & -8 \\ 2 & -1 & 2 \\ 5 & 4 & 6 \end{bmatrix}. \]
Verify that \(v_1 = (1,0,-1), \ v_2 = (2,1,-2), \) and \(v_3 = (0,-2,1)\) are eigenvectors of \(A\). Determine the associated eigenvalues \(\lambda_1, \lambda_2, \) and \(\lambda_3\). [Hint: Use the definition rather than the formula for determining eigenvalues!]

Q4: Suppose that the equation \(A\mathbf{v} = \lambda\mathbf{v}\) has real eigenvalue/eigenvector pairs \(\{\lambda_1, \mathbf{v}_1\}\) and \(\{\lambda_2, \mathbf{v}_2\}\). If we define the matrices
\[ P = [ \mathbf{v}_1 | \mathbf{v}_2 ], \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \]
we may combine \(A\mathbf{v}_1 = \lambda_1\mathbf{v}_1\) and \(A\mathbf{v}_2 = \lambda_2\mathbf{v}_2\) to write \(AP = PD\). This in turn allows us to write \(A = PDP^{-1}\). Since the matrix \(D\) is diagonal (i.e. has entries only along the main diagonal), this representation is called the diagonalization of \(A\).

For the matrix \[ \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} \] considered in 2(a), find the matrices \(P, D,\) and \(P^{-1}\), and verify that the formula \(A = PDP^{-1}\) works.
**BONUS:** A matrix is called a rotation matrix if it has the form

\[
R = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

for some fixed \( \theta \in [0, 2\pi) \). The matrix \( R \) has the property that, for a given vector \( \mathbf{v} \in \mathbb{R}^2 \) and angle \( \theta \), the vector \( \mathbf{w} = A\mathbf{v} \in \mathbb{R}^2 \) is the vector \( \mathbf{v} \) rotated an angle \( \theta \) in the counter-clockwise direction.

Show (or verify) that \( R \) has the following eigenvalue and eigenvector pairs

\[
\lambda_1 = \cos(\theta) + \sin(\theta)i, \quad \mathbf{v}_1 = (i, 1) \\
\lambda_2 = \cos(\theta) - \sin(\theta)i, \quad \mathbf{v}_2 = (-i, 1).
\]

**[Hint:** You may derive the pairs directly; however, it may be quicker to verify them through the definition.]**