Section 1: Limits

For multivariate functions $z = f(x, y)$, the intuition for limits is exactly the same as it was for the single-variable functions $y = f(x)$. As our input gets “closer” and “closer” to a point in the domain, we would like the output to get “closer” and “closer” to a particular value. Sometimes we were even able to define limits at points where the output was not even defined.

Formally, we define the following.

**Definition 1**

We say that

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $(x, y) \in D(f)$ and

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$  

**Note:** The term $\sqrt{(x - a)^2 + (y - b)^2}$ can also be written $|x - a|$ where $x = (x, y)$ and $a = (a, b)$ in vector notation. This form generalizes easily to higher dimensions.

It is worth emphasizing that, in order to satisfy Definition 1, we need the limit to be close for all points in a neighborhood. In particular, it is not enough to find a single path along which $f(x, y)$ gets closer to $L$ as we follow the path closer to $(a, b)$. A consequence of this is that it is often more difficult to prove that a limit exists than it is to prove that it does not exist.
Example 1

Show that the following multivariate function does not have a limit at $(0, 0)$:

$$f(x, y) = \frac{2xy}{x^2 + y^2}.$$

Solution: We discovered earlier that the level curves $f(x, y) = C$ consisted of straight lines through $(0, 0)$ of the form

$$y = \left(1 \pm \sqrt{1 - \frac{C^2}{C}}\right)x.$$

This raises the interesting dilemma of what exactly happens at $(0, 0)$. The function is undefined there, so we have to ask the question of what happens as we approach in a limit.

The level curve analysis conducted previously suggests the the limit does not exist. This is because, as we travel along the different lines toward $(0, 0)$, we do not approach the same value—rather, we can choose a line to approach any value between $-1$ and $1$, corresponding to the admissible values of $C$.

To be rigorous, however, we must turn this into a mathematical argument. To simulate the effect of traveling along straight lines, we take $y = tx$ where $t \in \mathbb{R}$ is arbitrary. This gives

$$f(x, y) = f(x, tx) = \frac{2tx^2}{x^2 + t^2x^2} = \frac{2t}{1 + t^2}.$$

This does not depend on $x$ or $y$, so that the value of the function is constant along lines $y = tx$. Since taking $x \to 0$ implies $y \to 0$, we have $(x, y) \to (0, 0)$ so

$$\lim_{x \to 0} f(x, y) = \lim_{(x, y) \to (0, 0)} f(x, y) = \frac{2t}{1 + t^2}.$$

Since this depends on the choice of line $y = tx$, it follows that the limit at $(0, 0)$ does not exist.
Example 2

Show that the limit at \((0, 0)\) is undefined for the following function:

\[
f(x, y) = \frac{x^4}{x^4 + y^2}.
\]

Solution: So far we only have one tool in our toolbox for showing that a limit is undefined: finding two paths of approach along which the limit at \((a, b)\) is not equal. In the earlier case, we chose the two paths to be straight lines. However, in this case we can see that setting \(y = tx\) gives

\[
\lim_{x \to 0} f(x, tx) = \lim_{x \to 0} \frac{x^4}{x^4 + t^2x^2} = 1.
\]

That is, regardless of the line chosen, the function always converges to 1.

It is tempting to conclude that the limit equals 1 but this does not follow. In order for the limit to exist, we require that all paths of approach given the same limit. It is not enough that some paths give the same limit, even an infinite number of paths.

In order to find a path of approach which yields a violation, we check the level curves. We have

\[
\frac{x^4}{x^4 + y^2} = C \implies x^4 = Cx^4 + Cy^2 \implies y = \pm \sqrt{\frac{1-C}{C}}x^2.
\]
In other words, the function \( f(x, y) \) is constant along parabolas. We notice, however, that all of these parabolas go through \((0, 0)\). This leads us to suspect that if we take a path along any one of these parabolas, we will get contradictory limits at \((0, 0)\).

Indeed, we set \( y = tx^2 \) to get

\[
\lim_{x \to 0} f(x, tx^2) = \lim_{x \to 0} \frac{x^4}{x^4 + t^2x^4} = \frac{1}{1 + t^2}.
\]

Since this is not the same for all parabolas \( y = tx^2 \), it follows that the limit at \((0, 0)\) does not exist.

Example 3

Prove that the limit at \((0, 0)\) for the following function is zero:

\[
f(x, y) = \frac{x^3}{x^2 + y^2}
\]

**Solution:** We know that \( f(0, 0) \) is undefined, so we need to consider the formal definition of the limit. Given an \( \epsilon > 0 \), we need to find a \( \delta > 0 \) such that \( 0 < \sqrt{x^2 + y^2} < \delta \) implies \( |f(x, y) - 0| = |f(x, y)| < \epsilon \).

We consider

\[
|f(x, y)| = \left| \frac{x^3}{x^2 + y^2} \right| = \left| \frac{x^2}{x^2 + y^2} \frac{x}{x^2 + y^2} \right| \leq \left| \frac{x^2 + y^2}{x^2 + y^2} \frac{x}{x^2 + y^2} \right| = |x| \leq \sqrt{x^2 + y^2} < \epsilon.
\]
In this argument, we have used the inequalities $x^2 \leq x^2 + y^2$ and $|x| \leq \sqrt{x^2 + y^2}$, which should be obvious.

This argument tells us that, if we pick $\sqrt{x^2 + y^2} < \epsilon$, then we necessarily will have $|f(x, y)| < \delta = \epsilon$. In other words, given an arbitrary $\epsilon > 0$, we can pick $\delta = \epsilon$ in order to satisfy the definition of the limit, and we are done.

Note: If this argument seemed difficult, you are certainly not alone. Proving the existence of limits from the definition is very difficult, even for professional mathematicians. The $\epsilon$-$\delta$ arguments of this type are precursors of the very rich (but often very challenging) area known as mathematical analysis, which is studied in Math 129A. The important take away from this example is to be able to apply the definition to examples of similar complexity to that given.

Section 2: Continuity

The notion of continuity for multivariable function is exactly the same as it was for single variable functions. Functions are continuous at a point if the limit is defined there and equals the value at the point. We have the following formal definition.
Definition 2

A function \( f(x,y) \) is **continuous at** \((a,b)\) if

\[
\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).
\]

A function \( f(x,y) \) is **continuous** if it is continuous at every point in its domain.

Just as in the single variable case, continuity is a measure of the **connectedness** of a function. There is now, however, the subtlety that the a multivariate function must be connected through a point \((x,y)\) **no matter which path we take through the point**.

**Note:** A multivariate function \( f(x,y) \) does not need to be defined at all points in a neighborhood of a point \((a,b)\) to be continuous at \((a,b)\). Any neighborhood of a point in \( x^2 + y^2 = 9 \) for the function \( f(x,y) = \sqrt{9 - x^2 - y^2} \) has most of its points lying **outside** of the domain. It is enough, however, that the limit be defined at points within this boundary.

We have the following continuity properties for multivariate functions. We will not prove them.

**Theorem 1**

Consider functions \( f(x,y) \) and \( g(x,y) \). Then:

1. If \( f(x,y) \) and \( g(x,y) \) are continuous at \((a,b)\), then \( h(x,y) = f(x,y) + g(x,y) \) is continuous at \((a,b)\).

2. If \( f(x,y) \) and \( g(x,y) \) are continuous at \((a,b)\), then \( h(x,y) = f(x,y) \cdot g(x,y) \) is continuous at \((a,b)\).

3. If \( f(x,y) \) and \( g(x,y) \) are continuous at \((a,b)\) and \( g(a,b) \neq 0 \), then \( h(x,y) = f(x,y)/g(x,y) \) is continuous at \((a,b)\).
Example 4

Determine the points where the following function is continuous:

\[ f(x, y) = \begin{cases} 
    \sin(x), & \text{for } x + y \geq 0 \\
    \sin(y), & \text{for } x + y < 0.
\end{cases} \]

Solution: We get for free that \( f(x, y) \) is continuous in \( x + y > 0 \) and \( x + y < 0 \) because the functions defined in any sufficiently small neighborhood of a point there are continuous. It is only along the line \( x + y = 0 \) that we need to check, since the function definition changes there.

For any continuous point \((x^*, y^*)\) along this line (i.e. \( y^* = -x^* \)), we need

\[
\lim_{(x,y) \to (x^*,y^*)} f(x, y)
\]

to hold true for both functional definitions (i.e. on both sides of \( x + y = 0 \)). We have for \( x + y \geq 0 \)

\[
\lim_{(x,y) \to (x^*,y^*)} \sin(x) = \sin(x^*)
\]

and for \( x + y < 0 \)

\[
\lim_{(x,y) \to (x^*,y^*)} \sin(y) = \sin(y^*).
\]

Along the line \( y^* = -x^* \) we have \( \sin(y^*) = \sin(-x^*) = -\sin(x^*) \). Setting the two functional definitions equal, we get

\[
\sin(x^*) = -\sin(x^*).
\]

This is clearly only satisfied for points \( x^* \) such that \( \sin(x^*) = 0 \), which gives

\[
x^* = k\pi, \quad k \in \mathbb{Z}
\]

(i.e. \( k \) in the integers). It follows that \( f(x, y) \) is continuous for \( x + y < 0 \), \( x + y > 0 \), and at points \((x, y) = (k\pi, -k\pi)\) where \( k \in \mathbb{Z} \).
Suggested Problems

1. Sketch the contour plot of the following functions, then use the level curves to show that the limit \( \lim_{(x,y)\to(0,0)} f(x,y) \) does not exist:

   (a) \( f(x,y) = \frac{x^2}{x^2 + y^2} \)

   (b) \( f(x,y) = \frac{xy^2}{x^2 + y^4} \)

   (c) \( f(x,y) = \frac{x^2 + xy - y^2}{x^2 - xy + y^2} \)

   (d) \( f(x,y) = \frac{y}{\sqrt{x^2 + y^2}} \)

2. Use the definition of the limit to prove that \( \lim_{(x,y)\to(0,0)} f(x,y) = 0 \) for the following functions:

   (a) \( f(x,y) = \frac{x^2 y}{x^2 + y^2} \)

   (b) \( f(x,y) = \frac{x^2 y^2}{x^2 + y^4} \)