Section 1: Partial Derivatives

We have considered how limits and continuity may be extended to multi-variate functions. We now consider the cornerstone of calculus: derivatives. Recall that we motivated differentiation for single-variable functions as the rate of change of the output (e.g. $y$) with respect to changes in the input (e.g. $x$). For multivariate functions $z = f(x, y)$, we will have to be careful, since there are two inputs.

To see how we resolve this complication, consider the following picture:

Imagine this is a hill and we are curious as to how quickly we would ascend/descend if we were to take a step forward. We would quickly see that the question only becomes sensible after choosing a direction (say red, blue, or green above). In fact, in such case, the problem reduces to that of a one-dimensional one directive.

The question then becomes how, given a surface $f(x, y)$, we might compute these rates of change. For simplicity, we will focus on the red and blue lines above, corresponding to only changes in the $x$ or $y$ direction respectively. This might correspond, for instance, to traveling only north/south or only east/west. We define the following.
Definition 1
The **partial derivatives** of a multivariate function $f(x, y)$ are given by

$$
fx(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
$$

$$
fy(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.
$$

If $(a, b) \in D(f)$ then the **partial derivatives of $f$ at $(a, b)$** are given by

$$
fx(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}
$$

$$
fy(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.
$$

**Note:** There are various notations used to denote partial derivatives of functions $z = f(x, y)$. The most common are

$$
\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \partial_x f(x, y) = fx(x, y) = D_x f
$$

$$
\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \partial_y f(x, y) = fy(x, y) = D_y f.
$$

We will generally use $fx(x, y)$ and $fy(x, y)$; however, there are very important circumstances where other notations are preferable.

That is, the partial derivative $fx(x, y)$ corresponds to only traveling in the $x$ direction, ignoring changes in the variable $y$. Similarly, the partial derivative $fy(x, y)$ corresponds to only traveling in the $y$ direction, ignoring changes in $x$. We may safely ignore the changes in one or more variables if they are constant along the direction of differentiation.

We compute partial derivatives with respect to a given variable in exactly the same manner as regular derivatives exception that we **treat all other variables as thought they were constants**. For instance, when we are evaluating the partial derivative with respect to $y$ of $f(x, y) = xy$, we evaluate

$$
\frac{\partial}{\partial y} [xy] = x
$$
just as though we were evaluating

\[ \frac{d}{dy} [2y] = 2. \]

Given this one adjustment, all of our established derivative rules still apply. We have the following.

**Theorem 1**

Consider multivariate functions \( f(x, y) \) and \( g(x, y) \), and a single-variable function \( h(u) \). Then:

1. **Product Rule:**
   \[
   \frac{\partial}{\partial x} [f(x, y)g(x, y)] = f_x(x, y)g(x, y) + f(x, y)g_x(x, y)
   \]

2. **Quotient Rule:**
   \[
   \frac{\partial}{\partial x} \left[ \frac{f(x, y)}{g(x, y)} \right] = \frac{f_x(x, y)g(x, y) - f(x, y)g_x(x, y)}{g(x, y)^2}
   \]

3. **Chain Rule:**
   \[
   \frac{\partial}{\partial x} [h(f(x, y))] = h'(f(x, y))f_x(x, y)
   \]

These only give the rate of change in the \( x \) and \( y \) direction, not an arbitrary direction (say, the green curve earlier). We will handle the more general directional case later in this course.

**Example 1**

Find the partial derivatives with respect to \( x \) and \( y \) of

\[ f(x, y) = xy^2. \]

**Solution:** To determine the partial derivative with respect to \( x \), we take the derivative of \( f(x, y) \) with respect to \( x \) holding \( y \) constant. We have

\[
\frac{\partial}{\partial x} [xy^2] = y^2.
\]
Conversely, to take the partial derivative with respect to \( y \), we take the derivative of \( f(x, y) \) with respect to \( y \) and hold \( x \) constant. We have
\[
\frac{\partial}{\partial y} [xy^2] = 2xy.
\]

**Example 2**

Find the partial derivatives with respect to \( x \) and \( y \) of
\[
f(x, y) = \ln(xy).
\]

**Solution:** We have
\[
\frac{\partial}{\partial x} \ln(xy) = \frac{1}{xy} y = \frac{1}{x}
\]
and
\[
\frac{\partial}{\partial y} \ln(xy) = \frac{1}{xy} x = \frac{1}{y}
\]
where we have computed \( \frac{1}{xy} \) as the derivative of \( \ln(xy) \) (for either \( x \) or \( y \)) and added, for instance, the extra term \( y \) because \( \frac{\partial}{\partial x} [xy] = y \) by the chain rule.

**Example 3**

Evaluate \( \frac{\partial z}{\partial x} \) for the following implicitly-defined multivariate function at the point \( (1, 0, 1) \):
\[
xyz + \frac{x}{z} = 1.
\]

**Solution:** Since the requested derivative only involves \( z \) and \( x \), we will hold \( y \) constant and treat the equation as an implicit equation of \( z \) and
we have:

\[ \frac{\partial}{\partial x} \left[ xyz + \frac{x}{z^2} \right] = \frac{\partial}{\partial x} [1] \]

\[ \Rightarrow yz + x\frac{\partial z}{\partial x} + \frac{1}{z} - \frac{x}{z^2} \frac{\partial z}{\partial x} = 0 \]

\[ \Rightarrow \frac{\partial z}{\partial x} \left( xy - \frac{x}{z^2} \right) = -yz - \frac{1}{z} \]

\[ \Rightarrow \frac{\partial z}{\partial x} = -\frac{z^2 (1 + y^2)}{z (xyz^2 - x)} = -\frac{z(1 + y^2)}{xyz^2 - x} \]

It follows that

\[ \frac{\partial z}{\partial x}(1,0,1) = -\frac{(1)(1 + (0))}{(0) - (1)} = 1. \]

Section 2: Higher Order Derivatives

For single-variable functions \( y = f(x) \), we also considered **second-order derivatives**. These derivatives allowed us to determine whether a function was concave up (accelerating) or concave down (decelerating).

For example, consider the following picture:

Along the blue and red lines—corresponding to the highlight curves holding the other variable constant—we can conduct all of our tools for the analysis of single-variable functions. For instance, we can see that the blue curve is concave up for the majority of the plotting interval, while the red curve is concave down. We expect, therefore, that the second derivative of the blue curve is positive while the second-derivative of the red curve is negative.
Formally, for a multivariable function $f(x, y)$, we define the following:

$$\frac{\partial^2}{\partial x^2} f(x, y) = \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} f(x, y) \right] = \frac{\partial}{\partial x} f_x(x, y) = f_{xx}(x, y).$$

The method of computing a second-order partial derivative is exactly the same as that for a first-order partial derivative. That is, we take the derivative with respect to $x$, holding $y$ constant.

A more interesting case arises, however, when we consider taking the second derivative with respect to a variable different than the first derivative. For example, consider the operation

$$\frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} f(x, y) \right].$$

Since the first partial derivatives are themselves functions of $x$ and $y$, it certainly makes sense to define such an operation. The interpretation of these mixed-order partial derivatives, however, will be more complicated, but we will see in a few lectures that they are extremely useful.

In general, we have

$$\frac{\partial^2}{\partial y \partial x} = \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} f(x, y) \right] = \frac{\partial}{\partial y} f_x(x, y) = f_{xy}(x, y)$$

and

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} f(x, y) \right] = \frac{\partial}{\partial x} f_y(x, y) = f_{yx}(x, y).$$

These definitions easily generalize to higher-order derivatives and multivariate functions with more than two independent variables. For example, for a function $f(x, y, z)$ we have

$$\frac{\partial^4}{\partial x^2 \partial z \partial y} f(x, y, z) = \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial z} \frac{\partial}{\partial y} f(x, y, z) = f_{yxxx}(x, y, z).$$

**Note:** It is important to remember that the order of the derivatives in the shortform is read from left to right in the subscripts. For instance, in the above example, $f_{yxxx}(x, y, z)$ means we take the partial derivative with respect to $y$, then $z$, then $x$, then $x$ again.
Example 4
Find the four second-order partial derivatives of \( f(x, y) = e^{2x+y} \).

Solution: We have
\[
    f_x(x, y) = 2e^{2x+y}, \quad \text{and} \quad f_y(x, y) = e^{2x+y}.
\]
It follows that
\[
    f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x} \left[ 2e^{2x+y} \right] = 4e^{2x+y},
\]
\[
    f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y} \left[ 2e^{2x+y} \right] = 2e^{2x+y},
\]
\[
    f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x} \left[ e^{2x+y} \right] = 2e^{2x+y},
\]
\[
    f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y} \left[ e^{2x+y} \right] = e^{2x+y}.
\]

Example 5
Find the four second-order partial derivatives of \( f(x, y) = x/y \).

Solution: We have
\[
    f_x(x, y) = \frac{1}{y}, \quad \text{and} \quad f_y(x, y) = -\frac{x}{y^2}.
\]
It follows that
\[
    f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x} \left[ \frac{1}{y} \right] = 0,
\]
\[
    f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y} \left[ \frac{1}{y} \right] = -\frac{1}{y^2}.
\]
\[
\begin{align*}
  f_{yx}(x, y) &= \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x} \left[ -\frac{x}{y^2} \right] = -\frac{1}{y^2} \\
  f_{yy}(x, y) &= \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y} \left[ -\frac{x}{y^2} \right] = \frac{2x}{y^3}.
\end{align*}
\]

Example 6

Determine \( f_{yxx}(x, y), f_{xyx}(x, y), \) and \( f_{xxy}(x, y) \) for \( f(x, y) = e^x \sin(y) \).

Solution: We have

\[
\begin{align*}
  f_x(x, y) &= e^x \sin(y), \quad \text{and} \quad f_y(x, y) = e^x \cos(y)
\end{align*}
\]

so that

\[
\begin{align*}
  f_{xx}(x, y) &= e^x \sin(y),
  f_{yx} = e^x \cos(y), \quad \text{and} \quad f_{xy} = e^x \cos(y).
\end{align*}
\]

We therefore have

\[
\begin{align*}
  f_{yxx}(x, y) &= \frac{\partial}{\partial x} f_{yx}(x, y) = \frac{\partial}{\partial x} [e^x \cos(y)] = e^x \cos(y) \\
  f_{xyx}(x, y) &= \frac{\partial}{\partial x} f_{xy}(x, y) = \frac{\partial}{\partial x} [e^x \cos(y)] = e^x \cos(y) \\
  f_{xxy}(x, y) &= \frac{\partial}{\partial y} f_{xx}(x, y) = \frac{\partial}{\partial x} [e^x \sin(y)] = e^x \cos(y).
\end{align*}
\]

Notice that, in all of the examples presented, the mixed partial derivatives are equal. That is, we always had \( f_{xy}(x, y) = f_{yx}(x, y) \), or in the final example \( f_{yxx}(x, y) = f_{xyx}(x, y) = f_{xxy}(x, y) \). If this were true in general, it would simplify the amount of work we have to do! We have the following result.

Theorem 2

Consider any higher-order partial derivatives of a function \( f(x, y) \) where the derivative of \( x \) has been taken \( m \) times, and the derivative with respect to \( y \) has been taken \( n \) times. If all the partial derivatives are continuous, then the two mixed partial derivatives are equal, and we
\[
\frac{\partial^{m+n}}{\partial x^m \partial y^n} f(x, y).
\]

Example 7

Determine \( f_{yyyyxx}(x, y) \) for \( f(x, y) = x \sin(\sqrt{ye^y}) \).

Solution: Our definition of partial differentiation tells us to take three derivatives with respect to \( y \), then two derivatives with respect to \( x \). We would quickly grow tired of taking derivatives with respect to \( y \), however, since the portion of \( f(x, y) \) containing \( y \) is complicated.

We can use the equality of mixed partial derivatives, however, to recognize that we can take the partial derivatives in any order we like. In particular, we can take the \( x \) derivatives before the \( y \) derivatives so we have

\[
f_{yyyyxx}(x, y) = f_{xxyyyy}(x, y).
\]

Since \( f_{xx}(x, y) = 0 \) it follows that \( f_{yyyyxx}(x, y) = 0 \).

Suggested Problems

1. Determine the first-order and second-order partial derivatives with respect to \( x \) and \( y \) of the following functions:

(a) \( f(x, y) = x^2 + 2xy \) \quad (e) \( f(x, y) = e^{\cos(x+2y)} \)
(b) \( f(x, y) = \frac{x}{x+y} \) \quad (f) \( f(x, y) = e^{x/y} \ln(x+1) \)
(c) \( f(x, y) = \frac{x+y}{x-y} \) \quad (g) \( f(x, y) = \arctan(x+y) + \ln(1+(x+y)^2) \)
(d) \( f(x, y) = \sin(xy) + \cos(xy) \) \quad (h) \( f(x, y) = x e^y \sin(xy) \)