Section 1: Tangent Planes

Recall that for single valued functions $y = f(x)$, in order to build a tangent line $T(x) = mx + b$ which agreed at the point $x = a$, we required agreement in two pieces of information: (a) $T(a) = f(a)$; and (b) $T'(a) = f'(a)$. That is, we need to pass through the same point and have the same slope.

We can immediately extend the earlier one-dimensional notion of a tangent line at a point to the two-dimensional case. In this case, we have a tangent plane rather than a tangent line, since we must specify the rate of change in two directions. We seek a plane of the form $T(x,y) = A + Bx + Cy$ which agrees at a point $(x_0, y_0)$ in the following three pieces of information:

1. $T(x_0, y_0) = f(x_0, y_0)$;
2. $T_x(x_0, y_0) = f_x(x_0, y_0)$; and
3. $T_y(x_0, y_0) = f_y(x_0, y_0)$.

With a little careful algebra, this system can be solved for the coefficients $A$, $B$, and $C$ in $T(x,y)$. We have:

- $T(x_0, y_0) = A + Bx_0 + Cy_0 = f(x_0, y_0)$
- $T_x(x_0, y_0) = B = f_x(x_0, y_0)$
- $T_y(x_0, y_0) = C = f_y(x_0, y_0)$.

We now have equations for $B$ and $C$, and can use these to backsubstitute for $A$ to get $A = f(x_0, y_0) - f_x(x_0, y_0)x_0 - f_y(x_0, y_0)y_0$. Substituting this into $T(x,y) = A + Bx + Cy$ gives the tangent plane or linear approximation formula

$$T(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$  (1)
Note: If the formula (1) reminds you of Taylor’s formula for single-variable functions, you are not alone. In fact, we can guarantee further agreement with the function $f(x, y)$ by considering second-order powers in our approximation $T(x, y)$ (i.e. adding terms like $x^2$, $xy$, and $y^2$) and then forcing the second-order derivatives to agree with $f(x, y)$.

**Example 1**

Determine the tangent plane to the following function at $(x_0, y_0) = (1/2, 0)$:

$f(x, y) = 1 - x^2 - y^2 + x^2y^2$.

**Solution:** To apply the formula, we need $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$. We have:

$$f(1/2, 0) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$$f_x(x, y) = -2x + 2xy^2 \implies f_x(1/2, 0) = -1.$$  

$$f_y(x, y) = -2y + 2x^2y \implies f_y(1/2, 0) = 0.$$  

It follows that the tangent plane is given by

$$T(x, y) = \frac{5}{4} - x.$$  

We can see in the following picture that this plane matches our intuition for what the tangent plane should do:
Example 2

Find the equation to the tangent plane to \( f(x, y) = x^2 + y^2 \) at the point \((2, -1)\). Use it to approximate the value of \( f(x, y) \) at \((2.1, -0.9)\) and give the error with the true value.

**Solution:** We need to find \( f(2, -1) \), \( f_x(2, -1) \), and \( f_y(2, -1) \). We have

\[
\begin{align*}
  f(2, -1) &= (2)^2 + (-1)^2 = 5 \\
  f_x(x, y) &= 2x \quad \Rightarrow \quad f_x(2, -1) = 4 \\
  f_y(x, y) &= 2y \quad \Rightarrow \quad f_y(2, -1) = -2.
\end{align*}
\]

It follows that the equation of the tangent line is

\[
T(x, y) = 5 + 4(x - 2) - 2(y + 1).
\]

This gives the approximation

\[
f(2.1, -0.9) \approx T(2.1, -0.9) = 5 + 4(0.1) - 2(0.1) = 5.2.
\]

We can calculate that the true value is

\[
f(2.1, -0.9) = (2.1)^2 + (-0.9)^2 = 5.22
\]

so that the error is

\[
|T(2.1, 0.9) - f(2.1, 0.9)| = |5.2 - 5.22| = 0.02.
\]
Example 3

Determine the point \((x_0, y_0)\) which has a tangent plane to the curve \(f(x, y) = xy\) which goes through the points \(T(0, 0) = -2\) and \(T(1, 2) = 2\).

**Solution:** The equation for the tangent plane to a point \((x_0, y_0)\) is

\[
T(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

We need to determine \(x_0\) and \(y_0\) given that we know \(T(0, 0) = -2\) and \(T(1, 2) = 2\). We have

\[
\begin{align*}
f(x_0, y_0) &= x_0y_0, \\
f_x(x, y) &= y \\f_y(x, y) &= x
\end{align*}
\]
so that \(T(x, y) = x_0y_0 + y_0(x - x_0) + x_0(y - y_0)\). Now we have

\[
\begin{align*}
T(0, 0) &= x_0y_0 + y_0((0) - x_0) + x_0((0) - y_0) \\
&= x_0y_0 - x_0y_0 - x_0y_0 = -x_0y_0 = -2
\end{align*}
\]
and

\[
\begin{align*}
T(1, 2) &= x_0y_0 + y_0((1) - x_0) + x_0((2) - y_0) = y_0 + 2x_0 - x_0y_0 = 2.
\end{align*}
\]

We can solve for \(y_0\) in the first equation to get \(y_0 = 2/x_0\), which can be plugged into the second equation to give

\[
\begin{align*}
\frac{2}{x_0} + 2x_0 - 2 &= 2 \\
2x_0 + 2x_0^2 &= 4x_0 \\
x_0^2 - 2x_0 + 1 &= (x_0 - 1)^2 = 0.
\end{align*}
\]

This implies that \(x_0 = 1\). This can be substituted back into any of the earlier equations to give \(y_0 = 2\). It follows that the point we must choose is \((x_0, y_0) = (1, 2)\), which corresponds to the tangent plane

\[
T(x, y) = 2 + 2(x - 1) + (y - 2).
\]
Suggested Problems

1. Determine the tangent plane to the following functions at the given points:

   (a) \[ f(x, y) = xy + x^3 \]
       \[ (x_0, y_0) = (1, -1) \]

   (b) \[ f(x, y) = \frac{1}{x^2 + 4y^2} \]
       \[ (x_0, y_0) = (0, 1) \]

   (c) \[ f(x, y) = \sin(x + y) \]
       \[ (x_0, y_0) = (\pi, 0) \]

   (d) \[ f(x, y) = e^{-\frac{x}{y}} \]
       \[ (x_0, y_0) = (0, 1) \]

2. Use the linear approximation at the given point \((x_0, y_0)\) to estimate the value of \(f(x, y)\) at the point \((x^*, y^*)\):

   (a) \[ f(x, y) = x^2 - xy + y^2 \]
       \[ (x_0, y_0) = (0, -1) \]
       \[ (x^*, y^*) = (0.1, -1.1) \]

   (b) \[ f(x, y) = \ln(x + y) \]
       \[ (x_0, y_0) = (1, 0) \]
       \[ (x^*, y^*) = (0.9, 0.1) \]