Consider a pendulum acting under the force of gravity (alternatively, an elongated spring obeying Hooke’s law). Suppose the rest position is $x = 0$ so that any displacement to the right corresponds to $x > 0$ while displacement to the left corresponds to $x < 0$.

If we move the pendulum to the right ($x > 0$), gravity acts against the pendulum to force it left ($F < 0$); conversely, if we move the pendulum to the left ($x > 0$), gravity acts against the pendulum to force it right ($F > 0$).

This gives rise to what is known as a **restoring force**. While it is clear that this force should act in the **opposite direction of the displacement**, it is not so clear what exact form it should take. It is common (and convenient!) to assume that the strength of the restoring force is **proportional to the displacement**. That is, we assume there is some $k > 0$ such that

$$F_{\text{restoring}} = -kx.$$  

In most realistic situations, of course, our pendulum would encounter more that simply a restoring force. The most obvious further force to add...
is a **damping force** corresponding to air resistance and/or friction from the pendulum’s hinge. If we imagine that \( x' = 0 \) corresponds to no velocity, \( x' > 0 \) corresponds to movement to the right, and \( x' < 0 \) corresponds to movement to the left, we imagine that the damping force should always acts in the **opposite direction as the velocity**. It will once again be convenient to assume that the strength of damping force is **proportional to the velocity**. That is, we assume there is some \( c > 0 \) such that

\[
F_{\text{damping}} = -cx'.
\]

We now want to incorporate these assumptions into a differential equation model for the pendulum. We will invoke Newton’s famous second law of classical mechanics \( F = ma \). We have that

\[
ma = mx''
\]

while

\[
F = F_{\text{restoring}} + F_{\text{damping}} = -kx - cx'.
\]

Our resulting differential equation is

\[
mx'' + cx' + kx = 0, \quad x(0) = x_0, \quad x'(0) = v_0.
\]  

We could derive the same differential equations, with a slightly different interpretation of the constants involved, by considering a mass-spring example obeying Hooke’s law.

**Note:** Even though (1) is a single equation, we require **two** initial conditions. We can realize this by considering the pendulum example. Consider looking at a snapshot of a pendulum at the resting position \( x = 0 \) and asking how the pendulum will move in the next instant. We should quickly realize that there are three possibilities:

1. The pendulum could truly have been **at rest** (i.e. it was not moving), in which case it will stay there.

2. The pendulum could have been **swinging to the right**, in which case it will continue to the right, lose speed, and eventually reverse (or swing over the top).

3. The pendulum could have been **swinging to the left**, in which case it will continue to the left, lose speed, and eventually reverse (or swing over the top).
In any case, we see that it is very important to consider not only the **position** of the pendulum at the time the snapshot was taken, but also the **velocity**—that is, we need two initial conditions.

### Example 1

For the values \( m = 1, c = 0, \) and \( k = \omega^2, \) corresponding to an undamped pendulum, rewrite the differential equation (1) as a system of DEs and sketch the vector field. Then determine the solution \( x(t) \) for \( x(0) = 0 \) and \( x'(0) = 1. \)

**Solution:** We are interested in the initial value problem

\[
x'' + \omega^2 x = 0, \quad x(0) = 0, \quad x'(0) = 1.
\]

We can rewrite this second-order differential equation as a system of two first-order differential equations with the variable substitutions

\[
\begin{align*}
x_1 &= x \\
x_2 &= x'.
\end{align*}
\]

This gives the following system in \( x_1 \) and \( x_2: \)

\[
\begin{align*}
x_1' &= x_2, \quad x_1(0) = 0 \\
x_2' &= -\omega^2 x_1, \quad x_2(0) = 1.
\end{align*}
\]

This is a linear system in \( x_1 \) and \( x_2. \) To sketch the vector field, we first determine the nullclines:

\[
\begin{align*}
x_1' &= 0 \implies x_2 = 0 \\
x_2' &= 0 \implies x_1 = 0.
\end{align*}
\]

This divides the \((x_1, x_2)\)-plane into the quadrants. We can easily determine the regions where \( x_1' > 0, \ x_1' < 0, \ x_2' > 0, \) and \( x_2' < 0 \) to get the following picture:
We also know how to solve this system. The coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

so that we have

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{bmatrix}.$$ 

It follows that

$$\det(A - \lambda I) = 0 \implies \lambda^2 + \omega^2 = 0$$

which implies that $\lambda = \pm \omega i$ so that $\alpha = 0$ and $\beta = \omega$. The corresponding complex eigenvectors can be then found. We have

$$(A - \omega i) \mathbf{v} = 0 \implies \begin{bmatrix} -\omega i & 1 \\ -\omega^2 & -\omega i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

This gives

$$\begin{bmatrix} -\omega i & 1 \\ -\omega^2 & -\omega i \end{bmatrix} \implies \begin{bmatrix} (-\omega i)(\omega i) & (1)(\omega i) \\ -\omega^2 & -\omega i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} \omega^2 & \omega i \\ -\omega^2 & -\omega i \end{bmatrix} \implies \begin{bmatrix} 1 \frac{1}{\omega} \\ 0 \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
It follows that we have
\[ \mathbf{v} = (i, -\omega) = (0, -\omega) + (1, 0)i \]
so that \( \mathbf{a} = (0, -\omega) \) and \( \mathbf{b} = (1, 0) \). It follows that the general solution is
\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \left( \begin{bmatrix} 0 \\ -\omega \end{bmatrix} \cos(\omega t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(\omega t) \right) \\
+ C_2 \left( \begin{bmatrix} 0 \\ -\omega \end{bmatrix} \sin(\omega t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(\omega t) \right)
\]
The initial conditions \( x_1(0) = 0 \) and \( x_2(0) = 1 \) give
\[
\begin{bmatrix} 0 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 0 \\ -\omega \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
It follows that \( C_1 = -\frac{1}{\omega} \) and \( C_2 = 0 \). Recalling that \( x(t) = x_1(t) \), it follows that we have
\[ x(t) = \frac{1}{\omega} \sin(\omega t). \]
This tells us that the pendulum oscillates left and right forever with a period of \( 2\pi/\omega \) and amplitude of \( \frac{1}{\omega} \). This oscillatory behavior should come as no surprise since we have neglected to incorporate any damping in the system.

Consider now incorporating some damping force. Based on our previous analysis, we probably do not want to completely solve the system by the same method. Fortunately, we will develop a simpler method for solving these systems momentarily. For now we notice that in order to determine the qualitative behavior of the system, we need only determine the eigenvalues of the system. The substitution \( x_1 = x, x_2 = x' \) gives the system
\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= -\frac{k}{m} x_1 - \frac{c}{m} x_2.
\end{align*}
\]
The coefficient matrix is
\[
A = \begin{bmatrix} 0 & 1 \\
-\frac{k}{m} & -\frac{c}{m} \end{bmatrix}
\]
We can quickly determine that
\[
\det(A - \lambda I) = 0 \implies -\lambda \left( -\frac{c}{m} - \lambda \right) + \frac{k}{m} = 0
\]
\[\implies m\lambda^2 + c\lambda + k = 0.\]
It follows that
\[
\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.
\]
We can see that this gives the same three cases we have already discussed. In particular, we notice that oscillations are only possible if \(c^2 - 4mk < 0\), since this is the condition for obtaining sines and cosines in the solution. For \(c^2 - 4mk > 0\), then the solution will consist of exponentials. Furthermore, since \(\sqrt{c^2 - 4mk} < c\), we have that the exponential will have a negative power. The somewhat surprising conclusion is that we may eliminate oscillations in our system by increasing the damping!

In general, we have the following classifications.

**Classifications:** The system (1) is said to be:

1. **undamped** if \(c = 0\),
2. **underdamped** if \(c^2 - 4mk < 0\),
3. **critically damped** if \(c^2 - 4mk = 0\), and
4. **overdamped** if \(c^2 - 4mk > 0\).

We will work out the example forms of the solutions momentarily. We start by consider second-order systems of differential equations in general.
Consider the following.

**Definition 1**

A **second-order linear** differential equation is given by the form

\[ y'' + p(x)y' + q(x)y = g(x) \]  \hspace{1cm} (2)

where \( y = y(x) \). The DE (3) is said to have **constant coefficients** if \( p(x) \) and \( q(x) \) are constants. It is said to be **homogeneous** if \( g(x) = 0 \).

**Note:** The form (2) is a direct generalization of first-order linear differential equations \( y' + p(x)y = q(x) \). The integrating factor solution method, however, will unfortunately not generalize to this new class of equations. In fact, equations (2) are not guaranteed to have closed-form solutions at all!

We will start our consideration of second-order differential equations with the easiest possible case: **linear, homogeneous, and constant coefficient**. Such a differential equation can be written

\[ ay'' + by' + cy = 0. \]  \hspace{1cm} (3)

We already know that such a differential equation can be rewritten as a linear system in two variables and then solved, but suppose we did not know that. We cannot separate the variables, or find an integrating factor, or find an obvious substitution which will reduce the differential equation to first-order. This exhausts our list of solution methods.

In fact, we will simply take an **educated guess**. We know that the first order system

\[ y' = ry \]

has the exponential solution \( y(x) = e^{rx} \), so we will guess that a solution of (3) has the form

\[ y(x) = e^{rx} \]
for some $r$ and see what happens. In the worst case scenario, even if this does not work out, we have not lost a great deal of time. It is easy to take derivatives of the exponential function!

Consider the following.

**Example 2**

Find a solution of the following second-order differential equation in the form $y(x) = e^{rx}$:

$$y'' - 5y' + 4y = 0. \quad (4)$$

**Solution:** We will guess that the solution has the form $y(x) = e^{rx}$. This gives

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

so that

$$y'' - 5y' + 4y = r^2 e^{rx} - 5re^{rx} + 4e^{rx}$$

$$= e^{rx}(r^2 - 5r + 4)$$

$$= e^{rx}(r - 1)(r - 4) = 0.$$  

The exponential is always positive, so we have either $r = 1$ or $r = 4$. It follows that

$$y_1(x) = e^x \quad \text{and} \quad y_2(x) = e^{4x}$$

are solutions of the differential equation. In fact, we can easily verify this. For $y_1(x) = e^x$, we have

$$y_1'' - 5y_1' + 4y = e^x - 5e^x + 4e^x = 0$$

and for $y_2(x) = e^{4x}$, we have

$$y_2'' - 5y_2' + 4y = 16e^{4x} - 5(4e^{4x}) + 4e^{4x} = 0.$$  

We might be initially surprised that we have two solutions to the equation, but we should not be. We have already seen that the general solution of a linear system in two variables is composed of two solutions. The corresponding pair of undetermined constants could be determined by using the initial conditions. Unsurprisingly, since we have two initial conditions for second-order equations, we will also have two independent solutions with undetermined coefficients.
In fact, this is a general property which applies to all homogeneous linear systems. We have the following result, which is known as the Principle of Superposition.

**Theorem 1**

Suppose that \( y_1(x) \) and \( y_2(x) \) are solutions of

\[
y'' + p(x)y' + q(x)y = 0.
\]

Then \( y = C_1 y_1 + C_2 y_2 \) where \( C_1, C_2 \in \mathbb{R} \) is a solution of (5).

**Proof**

Since \( y_1(x) \) and \( y_2(x) \) are solutions of (5), it follows that

\[
y_1'' + p(x)y_1' + q(x)y_1 = 0 \quad \text{and} \quad y_2'' + p(x)y_2' + q(x)y_2 = 0.
\]

We will now check if \( y(x) = C_1 y_1(x) + C_2 y_2(x) \) is a solution of (5). Note first of all that we have

\[
y'(x) = C_1 y_1'(x) + C_2 y_2'(x) \quad \text{and} \quad y''(x) = C_1 y_1''(x) + C_2 y_2''(x).
\]

On the left-hand side of (5), we therefore have

\[
y'' + p(x)y' + q(x)y
\]

\[
= [C_1 y_1'' + C_2 y_2''] + p(x) [C_1 y_1' + C_2 y_2'] + q(x) [C_1 y_1 + C_2 y_2]
\]

\[
= C_1 [y_1'' + p(x)y_1' + q(x)y_1] + C_2 [y_2'' + p(x)y_2' + q(x)y_2]
\]

\[
= 0
\]

where the last line follows from (6). Connecting the first line to the final one, we have that \( y'' + p(x)y' + q(x)y = 0 \) so that \( y(x) \) is a solution of (5), and we are done.

**Note:** The principle of superposition does not require the equation to have constant coefficients. It is easy, however, to find violations of Theorem 1 for equations which are not linear or not homogeneous.
For example, the nonlinear (but homogeneous) DE $y' - 2y^{1/2} = 0$ has the solution $y(x) = x^2$ but $\tilde{y}(x) = Cx^2$ is not a solution for $C \neq \pm 1$.
Similarly, the nonhomogeneous (but linear) DE $y' - y = e^x$ has the solution $y(x) = xe^x$ but $\tilde{y}(x) = Cxe^x$ is not a solution for $C \neq 1$.

The principle of superposition clarifies our earlier concern about solutions to (1) and (3). We are not completely done, however. Consider guessing the solution form $y(x) = e^{rx}$ for the general form (3). We have

$$ay'' + by' + cy = 0 \implies e^{rx}(ar^2 + br + c) = 0 \implies ar^2 + br + c = 0.$$  

If we cannot factor this expression, we need to use the quadratic formula. This gives

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (7)$$

We know how obtain the two fundamental solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ for quadratics with two real-valued roots. When the roots are complex or repeated, however, we will need alternative solution forms. We have the following.

**Theorem 2**

The general solution of the second-order DE (3) is given by the following:

1. If $b^2 - 4ac > 0$ the general solution is $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ where $r_1$ and $r_2$ are the two distinct roots of (7).
2. If $b^2 - 4ac = 0$ the general solution is $y(x) = C_1 e^{rx} + C_2 xe^{rx}$ where $r$ is the single repeated root of (7).
3. If $b^2 - 4ac < 0$ the general solution is

$$y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$$

where $r = \alpha + \beta i$.

These forms should seem familiar—they are exactly the forms obtained in the solutions of linear systems of differential equations. It can be quickly checked that the three cases for the solution forms in Theorem 2 correspond to the three cases for the eigenvalues of the corresponding system. The solution method of guessing the form $y = e^{rx}$, however, is generally much quickly (although less powerful) than performing the linear algebra analysis.
Consider the following examples.

**Example 3**

Find the general solution of $4y'' + 12y' + 9y = 0$. Then find the particular solution for $y(0) = 2$ and $y'(0) = 0$.

**Solution:** We guess the solution form $y(x) = e^{rx}$. This gives

$$4y'' + 12y' + 9y = e^{rx}(4r^2 + 12r + 9) = e^{rx}(2r + 3)^2 = 0.$$ 

It follows that we only have a solution if $r = -3/2$. Since this is a repeated root, we are in Case 2 and the general solution is given by

$$y(x) = C_1 e^{-(3/2)x} + C_2 xe^{-(3/2)x}.$$ 

To solve for the particular solution, we compute

$$y'(x) = -\frac{3}{2} C_1 e^{-(3/2)x} + C_2 e^{-(3/2)x} x - \frac{3}{2} C_2 xe^{-(3/2)x}.$$ 

The conditions $y(0) = 3$ and $y'(0) = 0$ give the system

$$C_1 = 2$$

$$-\frac{3}{2} C_1 + C_2 = 0.$$ 

We can quickly solve this to get $C_1 = 2$ and $C_2 = 3$. It follows that the particular solution is $y(x) = 2e^{-(3/2)x} + 3xe^{-(3/2)x}$:

![Graph of the solution](image.png)
Example 4

Find the general solution of $y'' + 2y' + 2y = 0$. Then find the particular solution for $y(0) = 1$ and $y'(0) = -1$.

Solution: We guess the solution $y(x) = e^{rx}$. This gives

$$y'' + 2y' + 2y = e^{rx}(r^2 + 2r + 2) = 0.$$  

The quadratic formula gives the solution

$$r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i.$$  

Since this a complex root, we are in case 3 and the general solution is

$$y(x) = C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x).$$  

To solve for the particular solution, we compute

$$y'(x) = -C_1 e^{-x} \cos(x) - C_2 e^{-x} \sin(x) - C_1 e^{-x} \sin(x) + C_2 e^{-x} \cos(x)$$  

$$= -C_1 e^{-x} (\cos(x) + \sin(x)) + C_2 e^{-x} (\cos(x) - \sin(x)).$$  

The conditions $y(0) = 1$ and $y(0) = -1$ gives the system

$$C_1 = 1$$  
$$-C_1 + C_2 = -1.$$  

It follows immediately that $C_1 = 1$ and $C_2 = 0$ so that the particular solution is $y(x) = e^{-x} \cos(x)$.
1. Determine the values of \( m, c, \) or \( k \) for which the following systems are critically damped (all derivatives with respect to \( t \)):

   (a) \( mx'' + 4x' + 4x = 0 \)  \hspace{1cm} (d) \( mx'' + 4x' + kx = 0 \)

   (b) \( 9x'' + cx' + 36x = 0 \)  \hspace{1cm} (e) \( 4x'' + cx' + x = 0 \)

   (c) \( x'' + x' + kx = 0 \)  \hspace{1cm} (f) \( mx'' + cx' + 25x = 0 \)

2. Rewrite the following as a system and then use the system formulation to solve for \( x(t) \):

   (a) \( 4x'' + 4x' + x = 0 \), \( x(0) = 1 \), \( x'(0) = -1 \)

   (b) \( 2x'' + 4x' + 4x = 0 \), \( x(0) = 2 \), \( x'(0) = 0 \)

3. Solve the following second-order linear differential equations (all derivatives with respect to \( x \)):

   (a) \( y'' - 7y + 10y = 0 \), \( y(0) = 3 \), \( y'(0) = 0 \)

   (b) \( 4y'' - 12y + 9y = 0 \), \( y(0) = 1 \), \( y'(0) = -1/2 \)

   (c) \( y'' + 4y + 5y = 0 \), \( y(0) = 1 \), \( y'(0) = 1 \)

   (d) \( 2y'' - 9y + 7y = 0 \), \( y(0) = 0 \), \( y'(0) = 5 \)