Instructions

1. Fill out this cover page completely.

2. Answer questions in the space provided, using scratch paper for rough work.

3. Show all the work required to obtain your answers.

4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

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1. Short Answer:

(a) Give an example of a first-order linear differential equation.

(b) State at least one condition on a differential equation \( M(x,y) \, dx + N(x,y) \, dy = 0 \) which guarantees the existence of an *integrating factor* which will make it exact.

[Note: Assume the equation is not exact to begin with!]

2. True/False:

(a) If a solution \( y(x) \) of \( y' = f(x,y) \) exists, it must be the unique solution through any initial condition \( y(x_0) = y_0 \) and it must be defined everywhere on the domain \(-\infty < x < \infty\). [True / False]

(b) The integrating factor for \( y' = \frac{1}{2x} y + e^x \) is \( \mu(x) = \frac{1}{\sqrt{x}} \). [True / False]

(c) A power homogeneous differential equation may always be transformed into a separable differential equation by the substitution \( v = y/x \). [True / False]

(d) For a differential equation \( y' = f(x,y) \), it is always possible to find an explicit solution \( y(x) \) by applying one of the methods described in class. [True / False]
3. Slope Fields and Solutions:

Consider the following initial value problem:

\[
\begin{align*}
\frac{dy}{dx} &= e^{-x} - y \\
y(0) &= y_0
\end{align*}
\]  

(a) Sketch the slope field of (1) in the \((x,y)\)-plane. Overlay a few solutions and conjecture as to the behavior of solutions as \(x \to \infty\). \([\text{Hint: Setting } y' = C \text{ gives } y = e^{-x} - C.\]

(b) Show that \(y(x) = (x + y_0)e^{-x}\) is a solution of (1) for all \(y_0 \in \mathbb{R}\). \([\text{Hint: Remember to check the initial condition!}\]

(c) Does the behavior of the solution as \(x \to \infty\) match your prediction in part (a)? Explain why or why not.
4. First-Order Differential Equations:

Solve the following first-order differential equations:

\[
\begin{align*}
(\text{a}) & \quad \frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}, \quad x > 0 \\
(\text{b}) & \quad \begin{cases}
(x - \ln(y)) \, dx + \left( y - \frac{x}{y} \right) \, dy = 0, \\
y(1) = 1
\end{cases}
\end{align*}
\]
5. Applications: (Inflow/Outflow)

Consider a filled mixing tank with a volume of 20 gallons. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water) mixture at a rate of 2 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallons per minute.

(a) Use the given information to derive a differential equation which models the amount of salt in the tank.

(b) Find the general solution of the differential equation derived in part (a).

(c) Suppose there is initially no salt in the tank. How much salt is in the tank after ten minutes? [Note: You do not need to evaluate to a decimal value, although, if it helps, $e^{-1} \approx 0.3678794412$.]
6. Applications: (Chemical Reactions)

Consider a chemical reaction $X \rightarrow Y$ occurring in a beaker with continuous inflow of $X$ and outflow of both $X$ and $Y$. Suppose that, after reparametrization, this gives the initial value problem

\[
\begin{align*}
\frac{dx}{dt} &= 2 - 2x, \quad x(0) = 1 \\
\frac{dy}{dt} &= x - y, \quad y(0) = 1
\end{align*}
\]  

(2)

where $x(t)$ and $y(t)$ are the concentrations of $X$ and $Y$, respectively, at time $t \geq 0$.

(a) Solve the IVP (2) for $x(t)$ and $y(t)$. [Hint: Solve for $x(t)$ before attempting the $y(t)$ equation!]

(b) Interpret the result in terms of the physical system. What is special about the initial conditions?
THIS PAGE IS FOR ROUGH WORK