Contour Plots for Study Guide

2(a)

\[ f(x, y) = \frac{y}{x^2 - 1} \]

Contours are

\[ \frac{y}{x^2 - 1} = C \quad \implies \quad y = C(x^2 - 1) \]

**Parabolas** with roots are \( x = -1 \) and \( x = 1 \). \( C \) controls the stretching. Note that every parabola goes through \((-1, 0)\) and \((1, 0)\) and hence the function does not have a limit there.
2(b)

\[ f(x, y) = x^2 - y^2 + 2y \]

Contours are

\[ x^2 - y^2 + 2y = C \quad \implies \quad \frac{x^2}{C - 1} - \frac{(y - 1)^2}{C - 1} = 1 \]

**Hyperbolas** centered at \( x = 0, y = 1 \); open in \( x \) for \( C > 1 \), open in \( y \) for \( C < 1 \)
3(a)

\[ f(x, y) = \frac{y}{1 - e^x} \]

Contours are
\[ \frac{y}{1 - e^x} = C \quad \implies \quad y = C(1 - e^x) \]

**Shifted exponentials** which all go through \( x = 0, \ y = 0 \). \( C \) controls the steepness and direction of opening (i.e. up or down).
3(b)

\[ f(x, y) = \frac{2x^2y}{x^4 + y^2} \]

Contours are

\[
\frac{2x^2y}{x^4 + y^2} = C \implies 2x^2y = Cx^4 + Cy^2 \implies Cy^2 - 2x^2y + Cx^4
\]

\[
\implies y = \frac{2x^2 \pm \sqrt{4x^4 - 4C^2x^4}}{2C} = \left( \frac{1 \pm \sqrt{1 - C^2}}{C} \right)x^2
\]

**Parabolas** where \( C \) controls the steepness. Note that \( C = 0 \) corresponds to \( x = 0 \) or \( y = 0 \), \( C = 1 \) gives \( y = x^2 \), and \( C = -1 \) gives \( y = -x^2 \).
3(c)

\[ f(x, y) = \frac{x^2 + y^2}{y} \]

Contours are

\[ \frac{x^2 + y^2}{y} = C \quad \Rightarrow \quad x^2 + y^2 = Cy \quad \Rightarrow \quad x^2 + (y - \frac{C}{2})^2 = \frac{C^2}{4} \]

Circles with center \((0, \frac{C}{2})\) and radius \(\frac{C}{2}\). Note that all such circles go through \((0, 0)\) and we may allow \(C < 0\).