Math 32, Fall 2015, Term Test II, Study Sheet
Calculus III

Date: Tuesday, October 27
Time: 10:30-11:45 a.m.
Lecture Section: 001
Instructor: Matthew Johnston

Surname (Family Name): ____________________________

Given Name: ____________________________

SJSU Student ID Number: ____________________________

Instructions

1. Fill out this cover page completely.

2. Answer questions in the space provided, using scratch paper for rough work.

3. Show all the work required to obtain your answers.

4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

FOR EXAMINERS’ USE ONLY

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1. Arc Length:
   (a) Consider the following vector function:
   \[ \mathbf{r}(t) = \langle t, 2t^2, \frac{8}{3}t^3 \rangle. \]
   (i) Determine the direction and magnitude of the vector derivative at \( t = 1 \).
   (ii) Determine the length of the curve from \( 0 \leq t \leq 1 \).
   (b) Consider the following vector function:
   \[ \mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle. \]
   (i) State the domain of \( \mathbf{r} \).
   (ii) Determine the direction and magnitude of the vector derivative at \( t = \ln(2) \).
   [Note: You do not need to simplify!]
   (iii) Determine the length of the curve from \( 0 \leq t \leq 1 \).
   (c) Reparametrize the following curves by arc length, starting from \( t = 0 \):
      (i) \( \mathbf{r}(t) = \langle 1 + 2t, 5 - t, 3t \rangle \)
      (ii) \( \mathbf{r}(t) = \langle \sin(t), 2t, \cos(t) \rangle \)
      (iii) \( \mathbf{r}(t) = \langle e^{3t} \sin(3t), e^{3t} \cos(3t), 1 \rangle \)

2. Multivariate Functions:
   (a) Consider the following multivariate function:
   \[ f(x, y) = \frac{y}{x^2 - 1}. \]
   (i) State the domain of \( f \).
   (ii) Sketch the contour plot of \( f \) in the \((x, y)\)-plane.
   (iii) Determine the direction and magnitude of steepest ascent at the point \((0, 2)\).
   (iv) Is the function continuous at the point \((1, 0)\)? Justify your answer.
   (b) Consider the following multivariate function:
   \[ f(x, y) = x^2 - y^2 + 2y. \]
   (i) Sketch the contour plot of \( f \) in the \((x, y)\)-plane.
   (ii) Determine the direction and magnitude of steepest ascent at the point \((0, 0)\).
   (iii) Determine the direction and magnitude of steepest ascent at the point \((0, -1)\).
   What happens at this point?
3. Limits and Continuity:

(a) Sketch the contour plot for the following functions, and then show that the limit as 
\((x, y) \to (0, 0)\) does not exist by determining at least two paths of approach which 
have different limits as \((0, 0)\):

(i) \(f(x, y) = \frac{y}{1 - e^x}\)

(ii) \(f(x, y) = \frac{2x^2y}{x^4 + y^2}\)

(iii) \(f(x, y) = \frac{x^2 + y^2}{y}\)

4. Partial Derivatives:

(a) Consider the following multivariate function:

\[ f(x, y) = 2xy + \sqrt{x + 4y}. \]

(i) Determine the following quantities: \(f_x(x, y), f_y(x, y)\), and \(f_{xx}(x, y), f_{xy}(x, y)\), and \(f_{yy}(x, y)\).

(ii) Determine the equation of the tangent plane at the point \((0,1)\). Use it to approximate the value of \(f(-0.01, 1.01)\).

(b) Consider the following multivariate function:

\[ f(x, y) = \frac{x^2 - y^2}{x + y}. \]

(i) Determine the following quantities: \(f_x(x, y), f_y(x, y)\), and \(f_{xx}(x, y), f_{xy}(x, y)\), and \(f_{yy}(x, y)\).

(ii) Determine the equation of the tangent plane at the point \((0,1)\). Use it to approximate the value of \(f(0.1, 0.9)\).

(c) Consider the following multivariate function:

\[ f(x, y) = e^{-x}(\cos(\frac{1}{2}y) + \sin(\frac{1}{2}y)). \]

(i) Determine the following quantities: \(f_x(x, y), f_y(x, y)\), and \(f_{xx}(x, y), f_{xy}(x, y)\), and \(f_{yy}(x, y)\).

(ii) Determine the equation of the tangent plane at the point \((0,\pi)\). Use it to approximate the value of \(f(0.01, \pi - 0.01)\).

(d) Show that \(\nabla(|\mathbf{r}|) = \frac{\mathbf{r}}{|\mathbf{r}|}\) where \(\mathbf{r} = (x, y, z)\).
5. Chain Rule:

(a) Suppose \( f(x, y, z) = x^2 + yz, \) \( x(s, t) = s + t, \) \( y(s, t) = s^2, \) and \( z(s, t) = st. \) Determine \( \frac{\partial f}{\partial s} \) and \( \frac{\partial f}{\partial t}. \)

(b) If \( z = f(x, y) \) where \( x = r \cos(\theta) \) and \( y = r \sin(\theta), \) show that

\[
\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2.
\]

(c) Suppose that the one-directional wave equation

\[
\frac{\partial u}{\partial t} = c \frac{\partial u}{\partial x}
\]

where \( u(t, x) \) holds. Show that \( s = x - ct \) gives

\[
\frac{du}{ds} = 0.
\]
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