We now turn our attention to computing double integrals

$$\text{Volume} = \int_R f(x, y) \, dx \, dy = \lim_{{n \to \infty}} \sum_{{i=1}}^{n} \sum_{{j=1}}^{n} f(x_i^*, y_j^*) \Delta x \, \Delta y$$

where $R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\}$.

We know that such an integral corresponds to the volume bound by the surface $f(x, y)$ and the $(x, y)$-plane above the rectangular region $R$. Recall that we could imagine the process of computing

$$\text{Area} = \int_a^b f(x) \, dx$$
as sweeping quickly from 0 to $f(x)$ as we travel from left to right. Effectively, we were computing the area by sweeping out rectangles with zero width as we traveled across the interval of integration.

It turns out that we can do exactly the same process for functions $f(x, y)$. That is, we can sweep out the area using pillars with zero width and length. The question is how to accomplish this given that the region $R$ is not order in the same way the interval $a \leq x \leq b$ is. That is, we cannot simply sweep from left to right. The resolution comes from recognizing that we can break the problem into *successive* one-dimensional problems. Consider the following intuition:

1. For each fixed value of $x$, we have $F(y) = f(x, y)$, which can be integrated from $c \leq y \leq d$.

2. This produces an area which depends on $x$, say $A(x)$, which can be integrated from $a \leq x \leq b$.

That is, we can sweep in the $y$ direction first, holding $x$ constant, and then sweep in the $x$ direction. Consider the following picture:
This process is intuitively pleasing, since we can see that the volume swept out is exactly the volume underneath the surface $f(x, y)$. It also prescribes an algebraic formula for computing double integrals. For compute the areas $A(x)$ for a fixed $x$, we have

$$A(x) = \int_c^d f(x, y) \, dy.$$  

Then, to compute the volume, we have

$$\text{Volume} = \int_a^b A(x) \, dx = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx.$$  

(1)

where, in the final representation, we implicitly understand that we take the integral with respect to $y$ before the integral with respect to $x$. That is, we work from the inside out. The formula (1) is called an **iterated integral**.

**Note:** As with partial differentiation, when computing iterated integrals (1), we treat uninvolved variables as constants. For example, in the inner integral, we keep $x$ constant, and in the outer integral we keep $y$ constant. This process is often called **partial integration**.

**Note:** We can also set up the integral in the other direction, where we first integrate in the $x$ direction (holding $y$ constant) to obtain areas $A(y)$, which can then integrate in the $y$ direction to obtain the volume.
This gives the formula

$$\text{Volume} = \int_c^d A(y) \, dy = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$  \hspace{1cm} (2)

This corresponds to the following picture:

---

**Example 1**

Evaluate the following iterating integrals:

(a) $\int_{-1}^1 \int_0^2 (2xy + 3y^2) \, dy \, dx$

(b) $\int_0^2 \int_{-1}^1 (2xy + 3y^2) \, dx \, dy$

**Solution:** For (a), we have

$$\int_{-1}^1 \int_0^2 (2xy + 3y^2) \, dy \, dx = \int_{-1}^1 \left[ xy^2 + y^3 \right]_{y=0}^{y=2} \, dx = \int_{-1}^1 (4x + 8 - 0) \, dx = \left[ 2x^2 + 8x \right]_{x=-1}^{x=1} = (2 + 8) - (2 - 8) = 16.$$
For (b), we have

\[
\int_0^2 \int_{-1}^1 (2xy + 3y^2) \, dx \, dy
\]

\[
= \int_0^2 \left[ x^2y + 3y^2 \right]_{x=-1}^{x=1} \, dy
\]

\[
= \int_0^2 (y + 3y^2) - (y - 3y^2) \, dy
\]

\[
= \int_0^2 6y^2 \, dy
\]

\[
= [2y^3]_{y=2}^{y=0}
\]

\[
= 16.
\]

Section 2: Fubini’s Theorem

It should be striking that, for Example 1, we obtained the same value regardless of the order of integration. This is intuitively pleasing, since we expect to sweep out the same volume regardless of which direction we build our areas. This properties is guaranteed (for most reasonable functions \( f(x, y) \)) by the following result, which is known as **Fubini’s Theorem**.

**Theorem 1**

If \( f(x, y) \) is continuous on \( R = \{(x, y) \mid a \leq x \leq b, \, c \leq y \leq d\} \) then

\[
\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

The justification of Fubini’s Theorem is well beyond the scope of this course—in fact, it is typically restricted to graduate-level courses in mathematics. We will see that this is an incredibly powerful result, especially when we begin consideration of integrals over non-rectangular regions.
Example 2

Use Fubini’s Theorem to evaluate the following double integral:

\[ \int_1^4 \int_0^1 \frac{x}{(x+y)^2} \, dy \, dx \]

**Solution:** To compute the integral as stated, we first integrate with respect to \( y \), treating \( x \) as thought it were a constant. If we prefer, we can simply use the substitution \( u = x + y \), so that \( du = dy \), and \( y = 0 \) and \( y = 1 \) imply that \( u = x \) and \( y = x + 1 \), respectively. We have

\[
\int_0^1 \frac{x}{(x+y)^2} \, dy = x \int_x^{x+1} \frac{1}{u^2} \, du \\
= x \left[ -\frac{1}{u} \right]_x^{x+1} \\
= x \left[ 1 - \frac{1}{x+1} \right] \\
= x \left[ \frac{(x+1) - x}{x(x+1)} \right] \\
= \frac{1}{x+1}.
\]

Notice that we have integrated \( y \) out of the equation, but \( x \) remains. It follows that we have

\[
\int_1^4 \int_0^1 \frac{x}{(x+y)^2} \, dy \, dx = \int_1^4 \frac{1}{x+1} \, dx \\
= [\ln(x+1)]_1^4 \\
= \ln(5) - \ln(2).
\]

Now that we have the swing of things down, we recall that we could have used Fubini’s Theorem to invert the order of integration. In this
\[
\int_1^4 \int_0^1 \frac{x}{(x+y)^2} \, dy \, dx \\
= \int_0^1 \int_1^4 \frac{x}{(x+y)^2} \, dx \, dy.
\]

We will have to be a little bit sneaky here, since it is not obvious how the substitution \( u = x + y \) will help us (it won’t). We will write

\[
\int_0^1 \int_1^4 \frac{x+y}{(x+y)^2} \, dx \, dy
= \int_0^1 \int_1^4 \frac{1}{x+y} \, dx \, dy + \int_0^1 \int_1^4 -\frac{y}{(x+y)^2} \, dx \, dy.
\]

This is allowed, since we can split double integrals in the same was as single integrals (fortunately!). We can now use our substitution or, if we are getting the hang of things, evaluate directly. We have

\[
\int_0^1 \left[ \ln(x+y) \right]_1^4 \, dy + \int_0^1 \left[ \frac{y}{x+y} \right]_1^4 \, dy
= \int_0^1 (\ln(y+4) - \ln(y+1)) \, dy + \int_0^1 \left( \frac{y}{y+4} - \frac{y}{y+1} \right) \, dy
= \int_0^1 (\ln(y+4) - \ln(y+1)) \, dy - \int_0^1 \left( \frac{3y}{(y+1)(y+4)} \right) \, dy
\]

Now we are really stuck! None of this evaluates easily. It turns out that it all can be done, but not without some tricky integration by parts (omitted). Putting absolutely everything together, we have that

\[
\left[ (y+4) \ln(y+4) - (y+1) \ln(y+1) - 4 \ln(y+4) + \ln(y+1) - 3 \right]_0^1
\]

\[
= 5 \ln(5) - 2 \ln(2) - 4 \ln(5) + \ln(2) - 4 \ln(4) + 4 \ln(4) = \ln(5) - \ln(2).
\]

So, after a lot of work, we have verified that the answers match. However, we should be wary about the about of work it has taken! In general, it can be much easier to integrate in one order than in the other. Unfortunately, there is no simple rule which tells us which directly is going to
be easier. We must \textbf{grow comfortable with double integration}, so that if we must determine the order of integration, we can check both directions quickly.

\textbf{Example 3}

Use Fubini’s Theorem to compute
\[
\int \int_{\mathcal{R}} xe^{xy} \, dA
\]
over the region \( \mathcal{R} = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq \ln(2)\} \).

\textbf{Solution:} Fubini’s Theorem tells us we can integrate in whichever direction we wish. Integrating with respect to \( y \) first gives
\[
\int \int_{\mathcal{R}} xe^{xy} \, dA = \int_0^1 \int_0^{\ln(2)} xe^{xy} \, dy \, dx
\]
\[
= \int_0^1 \left[ e^{xy} \right]_{y=0}^{y=\ln(2)} \, dx
\]
\[
= \int_0^1 \left( e^{\ln(2)x} - 1 \right) \, dx
\]
\[
= \left[ \frac{e^{\ln(2)x}}{\ln(2)} - x \right]_{x=0}^{x=1}
\]
\[
= \frac{2}{\ln(2)} - 1 - \frac{1}{\ln(2)}
\]
\[
= \frac{1}{\ln(2)} - 1.
\]

For completeness, consider integrating with respect to \( x \) first. Since \( f(x, y) = xe^{xy} \) has no discontinuities, Fubini’s Theorem guarantees we
will obtain the same result. We have:

\[
\int_0^{\ln(2)} \int_0^1 xe^{xy} \, dx \, dy = \int_0^{\ln(2)} \left[ \frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy} \right]_0^1 \, dy = \int_0^{\ln(2)} \left( \frac{e^y}{y} - \frac{e^y}{y^2} + \frac{1}{y^2} \right) \, dy
\]

where we have integrated by parts with \( u = x \) and \( dv = e^{xy} \). In order to continue, we have to be very careful because \( y = 0 \) is undefined for the remaining function to be integrate. That is, we now have an improper integral. We will test the lower bound to \( y = a \) and take \( a \to 0 \) later. We have

\[
= \lim_{a \to 0} \left[ \int_a^{\ln(2)} \left( \frac{e^y}{y} - \frac{e^y}{y^2} + \frac{1}{y^2} \right) \, dy \right]
\]

where we have used the substitution \( v = e^y/y \) to evaluate the first integral. We finally have

\[
= \frac{1}{\ln(2)} - \lim_{a \to 0} \left[ \frac{e^a}{a} - \frac{1}{a} \right]
\]

\[
= \frac{1}{\ln(2)} - 1
\]

where we have use L'Hopital’s rule to evaluate the “0/0” limit

\[
\lim_{a \to 0} \frac{e^a}{a} = \lim_{a \to 0} \frac{e^a}{1} = 1.
\]

We should be pleased that the two approaches have produced the same result of \( 1/\ln(2) - 1 \); however, this example offers a very important lesson: **one direction of integration may be significantly easier than the other!** In case, we were able to integrate first with respect to \( y \) directly. When integrating with respect to \( x \), we had to use integration parts, a non-trivial substitution, and an improper integral.
Suggested Problems

1. Evaluate the following iterated integrals and then verify that Fubini’s Theorem holds (i.e. exchange the order of integration and evaluate):

   (a) \( \int_0^1 \int_0^1 (x^2 + y^2) \, dy \, dx \)

   (b) \( \int_{-1}^2 \int_0^1 (xy - x^2 + y^2) \, dy \, dx \)

   (c) \( \int_0^3 \int_0^\pi x \sin(y) \, dy \, dx \)

   (d) \( \int_0^1 \int_{-1}^0 xye^{x-y} \, dy \, dx \)

   (e) \( \int_0^1 \int_0^2 \frac{xy}{1 + x^2} \, dy \, dx \)

   (f) \( \int_1^2 \int_1^2 \frac{1}{x + y} \, dy \, dx \)

2. Determine the volume of the shape bound by the surface \( f(x, y) = (x^2 + 1) \cdot (y^2 + 1) \) and the \((x, y)\)-plane. [Note: To determine the region \( R \), determine where \( f(x, y) = 0 \)]