Math 133A, November 3:
Method of Undetermined Coefficients

Section 1: Undetermined Coefficients

The question now becomes how we find the particular solution \( y_p(x) \). There are two primary methods for such a problem. The first is called the method of undetermined coefficients, and will be an intuitive extension of what we have been doing to date—that is, we will be guessing the solution form.

The second method is called variation of parameters and does not require any guessing. It will, however, require a significantly greater amount of work and some potentially tricky integration—even when we could simply guess the solution. We will largely omit studying variation of parameters, except to note that it can be seen as an extension of the methods we learned when studying nonhomogeneous systems of first-order linear differential equations.

Consider a general differential equation of the form

\[
ay'' + by' + c = g(x)
\]  

where we want to determine the particular solution \( y_p(x) \). Notice that the LHS of (1) involves just \( y \) and its derivatives, while the RHS contains a known functions of \( x \). What we need is a form of \( y_p(x) \) which can be differentiated to give a function of the form of \( g(x) \). Notice that

\[
\frac{d}{dx}[\text{polynomial}] = \text{polynomial}
\]

\[
\frac{d}{dx}[\text{exponential}] = \text{exponential}
\]

\[
\frac{d}{dx}[\text{sine and/or cosine}] = \text{sine and/or cosine}.
\]

This suggests that, if \( g(x) \) is a polynomial \( y_p(x) \), if \( g(x) \) is exponential we should have an exponential \( y_p(x) \), and if \( g(x) \) is trigonometric we should have a trigonometric \( y_p(x) \). This suggests the following steps for solving a differential equation of the form (1):
Algorithm 1

1. Find the general solution \( y_c(x) = C_1 y_1(x) + C_2 y_2(x) \) of the homogeneous equation
\[
ay''_c + by'_c + cy_c = 0.
\]

2. Select a trial function \( y_p(x) \) according to the following:
   (a) If \( g(x) \) contains \( x^n \) then use
   \[
y_p(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0.
   \]
   (b) If \( g(x) \) contains \( e^{rx} \) then use
   \[
y_p(x) = Be^{rx}.
   \]
   (c) If \( g(x) \) contains \( \sin(ax) \) or \( \cos(ax) \) then use
   \[
y_p(x) = A \cos(ax) + B \sin(ax).
   \]

3. Substitute the trial function \( y_p(x) \) into
\[
ay''_p + by'_p + cy_p = g(x)
\]
and solve for the undetermined coefficients in \( y_p(x) \) (i.e. solve for \( A, B, A_0, A_1, \ldots \))

4. If relevant, use the initial conditions to solve for the undetermined constants in the general solution \( y(x) = y_c(x) + y_p(x) \).

Note: We may need to use combinations of these functions. For example, if we have \( g(x) = e^x \sin(x) \), we need to use \( y_p(x) = Ae^x \cos(x) + Be^x \sin(x) \). If we have \( g(x) = x^2 e^{-x} \), we would need to use \( y_p(x) = (Ax^2 + Bx + C)e^{-x} \), etc. Also note that the arguments inside the trigonometric and exponential terms are also important. For instance, the forcing term \( g(x) = \sin(x) + \cos(2x) \) requires the trial function \( y_p(x) = A \cos(x) + B \sin(x) + C \cos(2x) + D \sin(2x) \).
Example 1

Find the general solution of the differential equation

\[ y'' + 4y = e^{-x} - 3x^3. \]

**Solution:** We need to first solve the homogeneous equation

\[ y'' + 4y = 0. \]

The guess \( y_c(x) = e^{rx} \) gives \( e^{rx}(r^2 + 4) = 0 \) so that \( r = \pm 2i \). It follows that

\[ y_c(x) = C_1 \cos(2x) + C_2 \sin(2x). \]

We now need to use a trial function \( y_p(x) \) with a suitable form that it could give \( e^{-x} - 3x^3 \) after differentiation. We try

\[
\begin{align*}
y_p(x) &= Ae^{-x} + Bx^3 + Cx^2 + Dx + E \\
y_p'(x) &= -Ae^{-x} + 3Bx^2 + 2Cx + D \\
y_p''(x) &= Ae^{-x} + 6Bx + 2C.
\end{align*}
\]

It follows that the differential equation gives

\[
\begin{align*}
y_p''(x) + 4y_p(x) \\
&= (Ae^{-x} + 6Bx + 2C) + 4(Ae^{-x} + Bx^3 + Cx^2 + Dx + E) \\
&= 5Ae^{-x} + 4Bx^3 + 4Cx^2 + (6B + 4D)x + (2C + 4E) \\
&= e^{-x} - 3x^3.
\end{align*}
\]

It follows that we need to satisfy

\[
\begin{align*}
5A &= 1 \\
4B &= -3 \\
4C &= 0 \\
6B + 4D &= 0 \\
2C + 4E &= 0.
\end{align*}
\]
It follows that we have \( A = \frac{1}{5}, \ B = -\frac{3}{4}, \ C = 0, \ D = \frac{9}{8}, \) and \( E = 0. \) The corresponding particular solution is
\[
y_p(x) = \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.
\]
The general solution is therefore
\[
y(x) = y_c(x) + y_p(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.
\]

Example 2

A spring with a mass of 1 kg is stretched 1 m by a force of 2 N and experiences damping of 3 N when the velocity is 1 m/s. Supposing there is an external force of \( g(t) = 2e^{-t}\sin(t) \) N and the spring is released from rest 1m from its resting position, determine the solution of the system.

**Solution:** The given information tells us that we have \( m = 1 \) kg, \( c = 3 \) N/(m/s), and \( k = 2 \) N/m, so that
\[
x'' + 3x' + 2x = 2e^{-t}\sin(t).
\]
We first consider the homogeneous system \( x'' + 3x' + 2x = 0. \) Guessing the solution \( x(t) = e^{rt} \) gives \( r^2 + 3r + 2 = (r + 2)(r + 1) = 0 \) so that \( r = -1 \) and \( r = -2. \) It follows that
\[
x_c(t) = C_1e^{-t} + C_2e^{-2t}.
\]
To find the particular solution \( x_p(t), \) we guess the trial form
\[
x_p(t) = Ae^{-t}\sin(x) + Be^{-t}\cos(x).
\]
This gives
\[
x_p'(t) = -(A + B)e^{-t}\sin(x) + (A - B)e^{-t}\cos(x)
\]
\[
x_p''(t) = 2Be^{-t}\sin(t) - 2Ae^{-t}\cos(t).
\]
It follows that we have
\[
x''_p + 3x'_p + 2x_p = \left[ 2Be^{-t} \sin(t) - 2Ae^{-t} \cos(t) \right] \\
+ 3 \left[ -(A + B)e^{-t} \sin(x) + (A - B)e^{-t} \cos(x) \right] \\
+ 2 \left[ Ae^{-t} \sin(x) + Be^{-t} \cos(x) \right] \\
= -(A + B)e^{-t} \sin(t) + (A - B)e^{-t} \cos(t) \\
= 2e^{-t} \sin(t).
\]

It follows that we have
\[
-A - B = 2 \\
A - B = 0.
\]
so that, from the second equation, \( A = B \), which implies \( A = B = -1 \) by the first equation. It follows that we have the general solution
\[
x(t) = C_1 e^{-t} + C_2 e^{-2t} - e^{-t} \sin(t) - e^{-t} \cos(t).
\]
We have the initial conditions \( x(0) = 1 \) and \( x'(0) = 0 \). We have
\[
x'(t) = -C_1 e^{-t} - 2C_2 e^{-2t} + 2e^{-t} \sin(t)
\]
so that
\[
x(0) = 1 = C_1 + C_2 - 1 \\
x'(0) = 0 = -C_1 - 2C_2.
\]
It follows from the second equation that \( C_1 = -2C_2 \) so that \( C_2 = -2 \) from the first. It follows that \( C_1 = 4 \) so that our particular solution is
\[
x(t) = 4e^{-t} - 2e^{-2t} - e^{-t} \sin(t) - e^{-t} \cos(t).
\]
It turns out that we have only partially answered the question of how to solve a general second-order system with constant coefficients. Consider finding the general solution of the differential equation

\[ y'' + 4y = \cos(2x) \quad (2) \]

with the method given by Algorithm 1.

The complementary function is easily determined to be \( y_c(x) = C_1 \cos(2x) + C_2 \sin(2x) \). We confidently guess the trial function

\[ y_p(x) = A \cos(2x) + B \sin(2x). \]

This gives

\[ y_p'(x) = -2A \sin(2x) + 2B \cos(2x) \]
\[ y_p''(x) = -4A \cos(2x) + 4B \sin(2x). \]

However, we can easily check that

\[ y_p''(x) + 4y_p(x) = -4A \cos(2x) + 4B \sin(2x) + 4(A \cos(2x) + B \sin(2x)) = 0. \]

We need to match constants so that this equals \( g(x) = \sin(2x) \) but the terms on the LHS have vanished. There are no constants left to solve for!

Something has gone seriously wrong, but after a moment of thought we realize that we should have expected this. The complementary function is \( y_c(x) = C_1 \cos(2x) + C_2 \sin(2x) \) so that the combination of functions in the given trial function had to vanish when it was substituted into the LHS of (2). **Algorithm 1 will never work for differential equations where the complementary function \( y_c(x) \) contains terms requires of the proposed trial form \( y_p(x) \)!

For such differential equations, the fix is to choose a different trial function. We have the following modification of step 2. of Algorithm 1.

\[ 2. \, ^* \text{ If the complementary solution } y_c(x) \text{ contains common terms with the required trial functions } y_p(x) \text{ given in step 2. of Algorithm 1, then select the corresponding portion of the trial function } y_p(x) \]

\[ \text{according to the following:} \]
\[(a^*) \quad y_p(x) = A_n x^{n+s} + A_{n-1} x^{n+s-1} + \cdots + A_1 x^{s+1} + A_0 x^s\]

\[(b^*) \quad y_p(x) = B x^s e^{rx}\]

\[(c^*) \quad y_p(x) = A x^s \cos(ax) + B x^s \sin(ax)\]

where \(s\) is the lowest power which produces a term which is independent of those in \(y_c(x)\).

**Note:** We will not offer a rigorous proofs of the forms in Algorithm 2 in this course. It should be noted that the notion of multiplying by the independent variable \(x\) to generate independent solutions is a common technique, and was used previously to generate the solutions \(y(x) = C_1 e^{rx} + C_2 x e^{rx}\) for differential equations with repeated roots.

**Example 3**

Find the general solution of

\[y'' + 4y = \cos(2x)\]

**Solution:** The complementary function was \(y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)\) so we are not allowed to use \(y_p(x) = A \cos(2x) + B \sin(2x)\) as a trial function. Instead, we must use

\[y_p(x) = A x \cos(2x) + B x \sin(2x)\]

This gives

\[y_p'(x) = A \cos(2x) + B \sin(2x) - 2Ax \sin(2x) + 2Bx \cos(2x)\]

\[y_p''(x) = 4B \cos(2x) - 4A \sin(2x) - 4Ax \cos(2x) - 4Bx \sin(2x)\]
Plugging into the DE gives

\[ y_p'' + 4y_p = 4B \cos(2x) - 4A \sin(2x) - 4Ax \cos(2x) - 4Bx \sin(2x) \\
+ 4(Ax \cos(2x) + Bx \sin(2x)) \\
= 4B \cos(2x) - 4A \sin(2x) \\
= \cos(2x). \]

It follows that we need \( A = 0 \) and \( B = 1/4 \) so that we have the particular solution

\[ y_p(x) = \frac{1}{4}x \sin(2x). \]

The general solution of the differential equation is therefore

\[ y(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{4}x \sin(2x). \]

### Suggested Problems

1. Use Algorithm 1 to solve the following initial value problems (all derivatives with respect to \( x \)):

   (a) \[ \begin{cases} y'' - 3y' + 2y = 2e^{3x} , \\
                   y(0) = -1 , \\
                   y'(0) = 0 \end{cases} \]

   (b) \[ \begin{cases} 2y'' + 5y' - 3y = 10 \sin(x) , \\
                   y(0) = 1 , \\
                   y'(0) = 0 \end{cases} \]

   (c) \[ \begin{cases} y'' - 2y' + 5y = 5x - 2 , \\
                   y(0) = 1 , \\
                   y'(0) = -2 \end{cases} \]

   (d) \[ \begin{cases} 3y'' + 7y' + 2y = 12e^x + 4x^2 , \\
                   y(0) = 0 , \\
                   y'(0) = 0 \end{cases} \]

2. Use Algorithm 1 with the modified step 2* to solve the following initial value problems (all derivatives with respect to \( x \)):

   (a) \[ \begin{cases} y'' + 3y' + 2y = e^x , \\
                   y(0) = 0 , \\
                   y'(0) = 0 \end{cases} \]

   (b) \[ \begin{cases} 4y'' + 9y = 12 \sin\left(\frac{3}{2}x\right) , \\
                   y(0) = 3 , \\
                   y'(0) = -1 \end{cases} \]
(c) \[ \begin{cases} y'' - 6y' + 9y = 6xe^{3x}, \\ y(0) = 1, \\ y'(0) = -1 \end{cases} \quad \text{(d) } \begin{cases} y'' + 4y' + 5y = e^{-2x} \sin(x), \\ y(0) = 0, \\ y'(0) = 0 \end{cases} \]