Section 1: Mechanical Systems

We now focus our attention on the physical pendulum/spring models which motivated our study of second-order equations:

\[ mx'' + cx' + kx = g(t) \]

where \( m \) was the mass, \( c \) was the damping constant, \( k \) was the restoring constant, and \( g(t) \) was the forcing function. We want to focus, in particular, on the effect of the forcing term \( g(t) \) on the behavior of the solution.

First, however, we investigate an alternative form of oscillatory solutions of the form \( C_1 \sin(\omega t) + C_2 \cos(\omega t) \). We have the following result, which allows us to write such terms in a single function.

Lemma 1

For any constants \( C_1, C_2, \) and \( \omega \), we have

\[ C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \cos(\omega t - \alpha) \]  \hspace{1cm} (1)

where

\[ A = \sqrt{C_1^2 + C_2^2}, \quad \alpha = \begin{cases} \arctan \left( \frac{C_2}{C_1} \right), & \text{if } C_1 \geq 0 \\ \arctan \left( \frac{C_2}{C_1} \right) + \pi, & \text{if } C_1 \leq 0. \end{cases} \]
Proof
From basic trigonometric identities, we have

\[ A \cos(\omega t - \alpha) = A \cos(\alpha) \cos(\omega t) + A \sin(\alpha) \sin(\omega t). \]

In order to satisfy (1), we need to satisfy

\[
\begin{align*}
A \cos(\alpha) &= C_1 \\
A \sin(\alpha) &= C_2.
\end{align*}
\] (2)

Squaring and adding the equations gives

\[ A^2(\cos^2(\alpha) + \sin^2(\alpha)) = C_1^2 + C_2^2 \implies A = \sqrt{C_1^2 + C_2^2}. \]

This gives the required value of the amplitude \( A \). To determine \( \alpha \), we divide the second equation of (1) by the first to get

\[
\frac{A \sin(\alpha)}{A \cos(\alpha)} = \frac{C_2}{C_1} \implies \tan(\alpha) = \frac{C_2}{C_1} \implies \alpha = \arctan\left(\frac{C_2}{C_1}\right).
\]

Notice, however, that the range of \( \arctan(\cdot) \) is \((-\pi/2, \pi/2)\). We cannot obtain angles outside of this range, which includes everything in quadrant II and quadrant III. We notice that the point \((C_1, C_2)\) is in these quadrants only if \( C_1 \leq 0 \). We must therefore correct these angles by a factor of \( \pi \) if \( C_1 \leq 0 \), and we are done.

The advantage of the form \( A \cos(\omega t - \alpha) \) is that the phase-shift of oscillation is clearly identified. Consider the following example.

Example 1
Consider a 2 kg weight attached to the end of a spring which requires a force of 8 Newtons to stretch one meter. Suppose the spring does not experience any damping. If the mass is initially stretched 2 meters to the right and released with an initial velocity of 2 meters per second to the right, find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form \( x(t) = A \cos(\omega t + \alpha) \).
Solution: The given information implies that $m = 2$, $k = 8$ and $c = 0$. This gives the model

$$2x'' + 8x = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $y(x) = e^{rx}$ gives

$$e^{rx}(2r^2 + 8) = 2e^{rx}(r^2 + 4) = 0$$

so that $r = \pm 2i$. It follows that the general solution has the form

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

To find the particular solution satisfying the initial conditions, we must compute

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t).$$

The initial conditions give

$$x(0) = 2 \implies C_1 = 2$$
$$x'(0) = 2 \implies 2C_2 = 2 \implies C_2 = 1.$$

It follows that the particular solution is

$$x(t) = 2 \cos(2t) + \sin(2t).$$

We want to put the solution in the form $x(t) = A \cos(\omega t - \alpha)$. From Lemma 1, we have $C_1 = 2$ and $C_2 = 1$ so that $A = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\alpha = \tan^{-1}(1/2) \approx 0.4636$. We can check that $(C_1, C_2) = (2, 1)$ is the first quadrant so that we do not need to adjust by a factor of $\pi$. It follows that the solution can be written

$$x(t) = \sqrt{5} \cos(2t - 0.4636):$$

\[\text{Graph of } x(t) = \sqrt{5} \cos(2t - 0.4636)\]
Section 2: Resonance

We now consider adding a forcing term to our governing equations. Once again, it will be useful to represent the solutions in a simplified form which consists of a single term. In this case, we will have multiple trigonometric functions, one corresponding to a fast mode of oscillation, and one corresponding to a slow mode of oscillation. We have the following result.

**Lemma 2**

For all constants $\omega_0$ and $\omega$, we have

$$
\cos(\omega_0 t) - \cos(\omega t) = 2 \sin(\alpha t) \sin(\beta t)
$$

where

$$
\alpha = \frac{1}{2} (\omega_0 + \omega), \quad \text{and} \quad \beta = \frac{1}{2} (\omega_0 - \omega).
$$

**Proof**

The trigonometric identities $\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$ and $\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$ can be subtracted from one another to give $2 \sin(A) \sin(B) = \cos(A - B) - \cos(A + B)$. If we take $A = \frac{1}{2} (\omega_0 + \omega)t$ and $B = \frac{1}{2} (\omega_0 - \omega)t$ we have the desired result.

**Example 2**

Consider the initial value problem

$$
\begin{cases}
    x'' + 4x = \cos(\omega t) \\
    x(0) = 0 \\
    x'(0) = 0
\end{cases}
$$

where $\omega \neq 2$. Write the solution in the form $x(t) = A \sin(\alpha t) \sin(\beta t)$. Comment on the behavior of the corresponding pendulum-spring system as $\omega \to 2$. 
**Solution:** We have already seen that the complementary function for this differential equation is

\[ x_c(t) = C_1 \cos(2t) + C_2 \sin(2t). \]

Since \( \omega \neq 2 \), we use the trial function \( x_p(t) = A \cos(\omega t) + B \sin(\omega t) \). This gives

\[ x_p''(t) = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t) \]

so that we have

\[ x_p'' + 4x_p = \left[-A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)\right] + 4 \left[A \cos(\omega t) + B \sin(\omega t)\right] \]

\[ = (4 - \omega^2)(A \cos(\omega t) + B \sin(\omega t)) \]

\[ = \cos(\omega t). \]

Since \( \omega \neq 2 \) implies \( \omega^2 \neq 4 \), it follows that \( A = 1/(4 - \omega^2) \) and \( B = 0 \) so that we have the general solution

\[ x(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4 - \omega^2} \cos(\omega t). \]

We now use the initial conditions \( x(0) = x'(0) = 0 \) to solve for \( C_1 \) and \( C_2 \). We have

\[ x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - \frac{\omega}{4 - \omega^2} \sin(\omega t) \]

so that, at \( t = 0 \), we have

\[ C_1 = -\frac{1}{4 - \omega^2} \]

\[ 2C_2 = 0 \]

which implies \( C_1 = -1/(4 - \omega^2) \) and \( C_2 = 0 \). It follows that the particular solution is

\[ x(t) = -\frac{1}{4 - \omega^2} \cos(2t) + \frac{1}{4 - \omega^2} \cos(\omega t) \]

\[ = \frac{1}{4 - \omega^2} (\cos(\omega t) - \cos(2t)). \]
In terms of simplification, this is pretty good, but we want to express the solution in the form \( x(t) = A \sin(\alpha t) \sin(\beta t) \). We therefore use Lemma 2 with \( \alpha = \frac{1}{2} (2 + \omega) \) and \( \beta = \frac{1}{2} (2 - \omega) \) to write the solution as

\[
x(t) = \frac{2}{4 - \omega^2} \sin \left( \frac{1}{2} (2 + \omega) t \right) \sin \left( \frac{1}{2} (2 - \omega) t \right).
\]

This was a lot of algebra, but we have obtained an incredibly insightful form of the solution. We now have the solution decomposed into two sine functions with different frequencies, corresponding to the difference in the natural and forcing frequencies. If \( \omega \) is near 2, there is a separation of time-scales in the two modes. We have that

1. The slow oscillatory mode has wavelength \( 4\pi/(2 - \omega) \). This mode can be thought of as an envelop which restricts all other modes (since all other modes must multiply through this function, so can only be as big as this slow mode allows it to be).

2. The fast oscillatory mode has wavelength \( 4\pi/(2 + \omega) \). This mode oscillates faster than the other mode but is restricted through each period by its slower counterpart.

3. Since sine is bounded by \(-1\) and \(1\), the maximal amplitude of the solution is \(2/(4 - \omega^2)\).

This raises a very interesting question: What happens as the forcing frequency is changed related to the fixed natural frequency of the system (i.e. the frequency the undamped pendulum or spring swings when left alone)? In particular, what happens as \( \omega \to 2? \)
We can consider this as $\omega$ approaches 2 from either side, since the separation of time-scales holds. We make the following observations:

1. The $\omega$ approaches 2, the wavelength of the slow mode *explodes* while the wavelength of the fast mode stays roughly the same. That is to say, the separation in time-scales intensifies in that the number of times the fast mode completes its cycle before the slow mode completes its cycle becomes unbounded.

2. The amplitude $2/(4 - \omega^2)$ also *explodes*. In fact, in the limit, we have that the amplitude is infinite.

![Graphs showing oscillations at different $\omega$ values](image)

Something seems to be going incredibly wrong in this example. We know that the solution oscillates with a fixed period so, as time goes on, the solution will make increasingly erratic jumps from the positive extreme to the negative extreme. In fact, the jumps approach infinite amplitude even as the period approaches a finite limit.

To understand what is happening, we will reconsider the physical motivation. Recall that 2 is the natural frequency term for the *underlying system*—that is, this is the frequency at which the body naturally oscillates if we simply let it go. Now imagine shaking the pendulum at a frequency which is is completely *in phase* with the natural rhythm of the body. In this case, every time the pendulum naturally wants to swing left, we give it
an extra push, and every time it wants to swing right, we give it an extra push in that direction, too. If we do this exactly in sync with the body’s natural rhythm, we imagine that the amplitude will grow. This is like pushing somebody on a swing. We get the most out of our effort if we wait to push when the swing is at the top of its arc, and push in the direction it is traveling.

We have discovered an interesting phenomenon which is a concern in many applications—namely, that small amplitude forcing functions may yield large amplitude responses when the forcing frequency is close to the natural frequency of the system. This phenomenon is called resonance. To complete the discussion, we consider the limiting case of \( \omega = 2 \).

**Example 3**

Solve the initial value problem

\[
\begin{aligned}
  x'' + 4x &= \cos(2t) \\
  x(0) &= x'(0) = 0.
\end{aligned}
\]

Comment on the behavior of the solution in the context of the physical set-up as a pendulum/spring system.

**Solution:** Notice that the complementary solution remains \( x_c(t) = C_1 \cos(2t) + C_2 \sin(2t) \) but that the trial function \( x_p(t) = A \cos(2t) + B \sin(2t) \) no longer works. Rather, we must use the trial function \( x_p(t) = At \cos(2t) + Bt \sin(2t) \). This gives

\[
x_p''(t) = 4B \cos(2t) - 4A \sin(2t) - 4At \cos(2t) - 4Bt \sin(2t).
\]

We therefore have

\[
x_p'' + 4x_p = [4B \cos(2t) - 4A \sin(2t) - 4At \cos(2t) - 4Bt \sin(2t)] \\
+ 4 [A \cos(2t) + Bt \sin(2t)] \\
= 4B \cos(2t) - 4A \sin(2t) \\
= \cos(2t)
\]

so that \( A = 0 \) and \( B = 1/4 \). It follows that

\[
x(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4} t \sin(2t).
\]
The initial conditions \( x(0) = 0 \) and \( x'(0) = 0 \) give \( C_1 = C_2 = 0 \) so that we have the simple solution

\[
x(t) = \frac{1}{4} t \sin(2t).
\]

Just as we expected, we have a solution which oscillates with increasing amplitude (as \( t \) grows). We have filled in the gap in our previous physical reasoning. Even though the solution methods were completely different, the limit of the previous solution approaches this resonate solution as \( \omega \to 2 \).

Before we carry this example too far, we should recognize that there are physical constraints which we have no considered in this model. In particular, we have not considered any damping, which would certainly become a significant concern as the pendulum begins to pick up speed. We have also neglected that a pendulum may swing over the top, and that a spring may simply break if it is compressed or overextended to a significant degree. To incorporate these effects would require introducing nonlinearities which are beyond the scope of this course to handle.

**Suggested Problems**

1. Consider a 9 kg weight attached to the end of a spring which requires a force of 4 Newtons to stretch one meter. Suppose the spring does not experience any damping. If the mass is initially stretched 1 meters to the right and released from rest, find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form \( x(t) = A \cos(\omega t + \alpha) \).

2. Determine the value of \( \omega_0 \) for which the following systems experience pure resonance, and then determine the solution \( x(t) \) to the initial
value problem.

\[ \begin{align*}
(\text{a}) & \begin{cases}
x'' + 4x = 2 \cos(\omega_0 t) \\
x(0) = 0 \\
x'(0) = 0
\end{cases} & \quad (\text{c}) & \begin{cases}
25x'' + 16x = 40 \sin(\omega_0 t) + 40 \cos(\omega_0 t) \\
x(0) = 0 \\
x'(0) = 0
\end{cases} \\
(\text{b}) & \begin{cases}
9x'' + 4x = \sin(\omega_0 t) \\
x(0) = 0 \\
x'(0) = 0
\end{cases} & \quad (\text{d}) & \begin{cases}
4x'' + 16x = 16 \sin(\omega_0 t) \\
x(0) = 2 \\
x'(0) = -1
\end{cases}
\end{align*} \]

3. Consider the following example:

\[ x'' + x' + x = \cos(\omega t). \]  

(3)

where \( \omega \) is as yet undetermined. That is to say, suppose we have \( m = 1 \) \( \text{kg} \), \( c = 1 \) \( \text{N}/(\text{m/s}) \) and \( k = 1 \) \( \text{N}/\text{m} \).

(a) Find the general solution of (3). \[ \textbf{[Hint:} \text{Note that we do not need to consider cases for } \omega \text{!}] \]

(b) By considering the limit as \( t \to \infty \), divide the solution from part (a) into two parts: a \textit{transient solution} \( x_{tr}(t) \) which goes to zero in the limit, and a \textit{steady periodic} solution \( x_{sp}(t) \) which does not. (In other words, write \( x(t) = x_{tr}(t) + x_{sp}(t) \).)

(c) Find the amplitude of the steady periodic function \( x_{sp}(t) \) found in part (c). \[ \textbf{[Hint:} \text{Consider writing the portion } x_{sp}(t) \text{ in the form } A \cos(\omega t - \alpha) \text{ but only find } A. \]

(d) At which value of \( \omega \) does \( A \) achieve its maximum? Interpret this value in terms of the physical system. In particular, how does it compare to the quasi frequency of the unforced system? \[ \textbf{[Hint:} \text{Take the derivative of } A \text{ with respect to } \omega \text{!]}

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4. Consider a pendulum/spring modeled by the following equation

\[ x'' + cx' + x = \cos(t) \]

where \( c \geq 0 \) is the unspecified damping coefficient. By using the method of Question #1(a-b), determine the amplitude of the steady periodic portion of the solution as a function of \( c \) and find the maximal amplitude. To what type of system does the maximal amplitude correspond?