Section 1: Inverse Laplace Transform

A key portion of what we will do is converting back from the Laplace transform world to standard functions of $x$. This naturally means we will have to invert the Laplace transform, i.e. we will have to compute

$$\mathcal{L}^{-1}\{F(s)\} = f(x).$$

Formally defining this operation would be incredibly tedious, but it is also going to be unnecessary. We already know the basic forms of a number of Laplace transforms. From these known forms, we can begin to form a catalogue of inverse Laplace transforms.

### Inverse Laplace Transforms

- $\mathcal{L}\{e^{ax}\} = \frac{1}{s-a} \implies \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax}$
- $\mathcal{L}\{x^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\} = x^{n-1}$
- $\mathcal{L}\{\sin(bx)\} = \frac{b}{s^2 + b^2} \implies \mathcal{L}^{-1}\left\{\frac{b}{s^2 + b^2}\right\} = \sin(bx)$
- $\mathcal{L}\{\cos(bx)\} = \frac{s}{s^2 + b^2} \implies \mathcal{L}^{-1}\left\{\frac{s}{s^2 + b^2}\right\} = \cos(bx)$

We will add to this list later, but it will be important to go through a few examples now to see how these examples proceed.
Example 1

Determine the inverse Laplace transform of

\[ F(s) = \frac{24}{s^4} - \frac{9}{s^2 + 9}. \]

**Solution:** It is important first of all to recognize the linearity of the Laplace transform also applies to the inverse Laplace transforms. That is to say, we have

\[ \mathcal{L}^{-1}\{F(s) + G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}. \]

It follows from this observation that, for the original problem, we have

\[ \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{24}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{9}{s^2 + 9}\right\}. \]

We have to be careful at this point. We need to recognize that the inverse Laplace transfer forms depend explicitly on constants. For instance, we recognize that the first term corresponds structure to the form required for

\[ \mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\} = x^{n-1} \]

with \( n = 4 \). We will have to make sure that, however, that the constant in the numerator is the correct one! If we have \( n = 4 \), we must have \( (n-1)! = 3! = 6 \) absorbed by the inverse Laplace transform.

We will also have to be careful with the form

\[ \mathcal{L}^{-1}\left\{\frac{b}{s^2 + b^2}\right\} = \sin(bx). \]

We recognize that \( b = 3 \) for our given form, so this much will be absorb by the inverse transformation. All told, we should recognize that we have

\[ \mathcal{L}^{-1}\{F(s)\} = 4\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - 3\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\} = 4x^3 - 3\sin(3x). \]
Example 2

Find the inverse Laplace transform of

\[ F(s) = \frac{8}{s^3 + 4s}. \]

Solution: Our first observation is that, not matter how we adjust the constant in the numerator, this does not fit readily into our given forms. So what can we do?

The key, which will be a common feature of inverting Laplace transforms, is that we can factor the denominator into

\[ F(s) = \frac{8}{s^3 + 4s} = \frac{8}{s(s^2 + 4)}. \]

This may seem like a modest gain, but it actually leads to a general method. We know from work on integrate that we can break terms like this into separate terms with simpler denominators. In this case, we know that we can write this as

\[ \frac{8}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} \]

for some constants \( A, B, \text{ and } C \) by using partial fraction decomposition. The forms on the right-hand side are easily recognized from your table of inverse transforms! So partial fraction decomposition is the key to resolving inverse transformations with complicated denominators.

In this case, we have

\[ 8 = A(s^2 + 4) + (Bs + C)s \quad \Rightarrow \quad (A + B)s^2 + Cs + (4A - 8) = 0. \]

Equating coefficients on both sides gives the system

\[ \begin{align*}
A + B &= 0 \\
C &= 0 \\
4A - 8 &= 0.
\end{align*} \]
It follows from the final constraint that $A = 2$ so that $B = -2$ by the first. Also, clearly $C = 0$ by the second. It follows that we have

$$
\mathcal{L}^{-1}\left\{\frac{8}{s(s^2 + 4)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2 + 4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} = 2 - 2\cos(2x).
$$

**Section 2: Shifted Laplace Transform**

A particular nice property of Laplace transforms is that it very easily handles shifts in the domain variable $s$. For instance, imagine trying to determine the inverse Laplace transform

$$
\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 1}\right\}.
$$

While we can certainly identify elements of the Laplace transforms of $\sin(x)$, it is not quite in the necessary form because of the domain shift $s - 1$ in the denominator.

The following result allows us to compute such inverse transformations.

**Lemma 1**

Suppose that $\mathcal{L}\{f(x)\} = F(s)$. Then $\mathcal{L}\{e^{cx}f(x)\} = F(s-c)$. Conversely, we have $\mathcal{L}^{-1}\{F(s-c)\} = e^{cx}f(x)$.

**Proof**

The proof is simpler than likely anticipated. By definition, we have

$$
\mathcal{L}\{e^{cx}f(x)\} = \int_0^\infty e^{-sx}e^{cx}f(x)\,dx = \int_0^\infty e^{-(s-c)x}f(x)\,dx = F(s-c).
$$
An important consequence of Lemma 1 is that it very quickly allows us to expand our catalogue of Laplace transformations.

**Inverse Laplace Transforms**

\[
\mathcal{L}\{e^{ax}x^n\} = \frac{n!}{(s-a)^{n+1}} \implies \mathcal{L}^{-1}\left\{\frac{(n-1)!}{(s-a)^n}\right\} = e^{ax}x^{n-1}
\]

\[
\mathcal{L}\{e^{ax}\sin(bx)\} = \frac{b}{(s-a)^2 + b^2} \implies \mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2 + b^2}\right\} = e^{ax}\sin(bx)
\]

\[
\mathcal{L}\{e^{ax}\cos(bx)\} = \frac{(s-a)}{(s-a)^2 + b^2} \implies \mathcal{L}^{-1}\left\{\frac{(s-a)}{(s-a)^2 + b^2}\right\} = e^{ax}\cos(bx)
\]

where \(s > a\).

For our above example, we may now quickly identify that the inverse transformation gives

\[
\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 1}\right\} = e^x\sin(x).
\]

**Example 3**

Determine the inverse Laplace transform of

\[
F(s) = \frac{s - 1}{s^2 - 4s + 5}.
\]

**Solution:** We need to determine

\[
\mathcal{L}^{-1}\left\{\frac{s - 1}{s^2 - 4s + 5}\right\}
\]

but do not recognize this immediately as being in one of our prescribed forms. Worst still, the bottom cannot be factored (over the real numbers, anyway) so that we cannot split the denominator.

Our only alternative is to *complete the square* in the denominator. This will be a general method, in fact. If we have an irreducible quadratic term, we must complete the square to get it in the standard for \((s – \)
In this case, we have

\[ s^2 - 4s + 5 = (s^2 - 4s + 4) - 4 + 5 = (s - 2)^2 + 1. \]

Now we are getting somewhere! We have

\[
\mathcal{L}^{-1} \left\{ \frac{s - 1}{s^2 - 4s + 5} \right\} = \mathcal{L}^{-1} \left\{ \frac{s - 1}{(s - 2)^2 + 1} \right\}.
\]

This is pretty good, but we are not out of the woods yet. The shifted sine and cosine forms require us to have some factor of either the shift (i.e. \( s - c \)) or the remaining term (i.e. \( b \)) in the numerator. We clearly have \( c = 2 \) and \( b = 1 \) but we do not have \( s - 2 \) or \( 1 \) by themselves in the numerator. Instead, we must create them. In this case, we can simply adjust to get what we need. If we subtract by a one in the numerator, we need to add by one. This gives

\[
\mathcal{L}^{-1} \left\{ \frac{s - 1 - 1 + 1}{(s - 2)^2 + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{s - 2}{(s - 2)^2 + 1} + \frac{1}{(s - 2)^2 + 1} \right\}.
\]

This is exactly we needed! We can immediately recognize these as the shift sine and cosine forms. After a little bit of work, we have been able to show that

\[
\mathcal{L}^{-1} \left\{ \frac{s - 1}{s^2 - 4s + 5} \right\} = e^{2x} \cos(x) + e^{2x} \sin(x).
\]
Suggested Problems

1. Determine the Laplace transform of the following functions:
   
   (a) \( f(x) = xe^x \)  
   (b) \( f(x) = x(1 + x)e^{3x} \)  
   (c) \( f(x) = e^{-2x} \sin(x) \)  
   (d) \( f(x) = e^x \cos(\pi x) \)

2. Determine the inverse Laplace transform of the following functions:

   (a) \( F(s) = \frac{1}{s^2} \)  
   (b) \( F(s) = \frac{9}{s^2 + 16} \)  
   (c) \( F(s) = \frac{1}{s^2 - s} \)  
   (d) \( F(s) = \frac{2}{s^2 + 2s + 2} \)  
   (e) \( F(s) = \frac{2}{(s + 5)^4} \)  
   (f) \( F(s) = \frac{3}{4s^3 + 5s^2 + s} \)  
   (g) \( F(s) = \frac{s}{s^2 - 4} \)  
   (h) \( F(s) = \frac{s^2 + 5}{s^3 - 2s^2 + 5s} \)