Section 1: Initial Value Problems

We may now combine all this groundwork toward our ultimate goal of solving differential equations. Consider the following example.

**Example 1**

Use the Laplace transform to solve the following initial value problem

\[ y'' - 3y' + 2y = 0, \quad y(0) = 0, y'(0) = 1. \]

**Solution:** We already know how to solve this IVP by guessing \( y(x) = e^{rx} \) and solving for \( r \). After substituting the initial conditions, this would give us that answer

\[ y(x) = -e^x + e^{2x}. \]

Our first step to solving this by the Laplace transform method is to transform the entire differential equation into the Laplace domain. We have

\[
\mathcal{L}\{y'' - 3y' + 2y\} = 0 \\
\Rightarrow \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0 \\
\Rightarrow [s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = 0
\]

where \( Y(s) \) is the Laplace transform of \( y(x) \). What is interesting here is that all of the derivatives have been absorbed into the initial conditions, and we know what these are. The only thing unsolved for here is \( Y(s) \),
which we may isolate:

\[ \Rightarrow [s^2Y(s) - 1] - 3sY(s) + 2Y(s) = 0 \]
\[ \Rightarrow (s^2 - 3s + 2)Y(s) = 1 \]
\[ \Rightarrow Y(s) = \frac{1}{s^2 - 3s + 2}. \]

This is great! We have now isolated the Laplace transform of the solution we want. It remains only to invert the transformation, and we have already performed this task several times. We will need to factor and perform partial fraction decomposition on the right-hand side. We have

\[ Y(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}. \]

We can multiply this across to get

\[ 1 = A(s-2) + B(s-1). \]

Setting \( s = 1 \) gives \( A = -1 \) and setting \( s = 2 \) gives \( B = 1 \). It follows that we have

\[ Y(s) = -\frac{1}{s-1} + \frac{1}{s-2}. \]

As expected, we can invert this to get

\[ y(s) = -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = -e^x + e^{2x}. \]

We should be very happy that we obtained the earlier expected answer, but we might wonder why we needed another method in the first place. After all, the standard method works just fine for this example. The answer will not be fully apparent just yet, but it is worth noting that the Laplace transform method is a one-step solution method. Our previous method required separate steps to hand homogeneities and initial conditions. The Laplace transform method directly incorporates these things into the method itself.

To see how non-homogeneities are incorporated in the method, consider the following problem.
Example 2

Use the Laplace transform method to solve

\[ y'' + 4y' + 4y = 3xe^{-2x}, \quad y(0) = 0, \quad y'(0) = 1. \]

Solution: We have

\[
\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{3xe^{-2x}\}
\]

\[
\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 4Y(s) = \frac{3}{(s + 2)^2}
\]

\[
\Rightarrow \quad (s + 2)^2Y(s) = 1 + \frac{3}{(s + 2)^2}
\]

\[
\Rightarrow \quad Y(s) = \frac{1}{(s + 2)^2} + \frac{3}{(s + 2)^4}.
\]

We have

\[
y(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s + 2)^4}\right\}
\]

\[
= \mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2}\right\} + \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{6}{(s + 2)^4}\right\}
\]

\[
= xe^{-2x} + \frac{x^3}{2}e^{-2x}.
\]

This form can be easily verified directly or by solving using the classical method, but requires separate steps to determine the complementary solution \(y_c(x)\), the particular solution \(y_p(x)\), and then to evaluate the initial conditions. By contrast, the Laplace transform method employed above was a one-step method.

Example 3

Solve the following differential equation using the Laplace transform method:

\[ y'' + 4y' + 3y = 10\cos(x), \quad y(0) = 1, \quad y'(0) = 2. \]
Solution: We take the Laplace transform of the entire equation to get

\[ \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 10\mathcal{L}\{\cos(x)\} \]

\[ \implies [s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 3Y(s) = \frac{10s}{s^2 + 1} \]

\[ \implies (s^2 + 4s + 3)Y(s) = \frac{10s}{s^2 + 1} + s + 6 \]

\[ \implies (s + 3)(s + 1)Y(s) = \frac{10s}{s^2 + 1} + s + 6. \]

If we choose not to find a common denominator on the right-hand side, we encounter the equation

\[ Y(s) = \frac{10s}{(s + 1)(s + 3)(s^2 + 1)} + \frac{s + 6}{(s + 1)(s + 3)}. \]

This is correct but it is undeniably bad. Neither of these terms are directly in a form where we can identify the inverse transform, and instead of performing partial fraction decomposition once, we will have to perform it twice (once for each term!). It is, in this case much easier to find a common denominator first. In this case, we have

\[ (s + 3)(s + 1)Y(s) = \frac{10s + (s + 6)(s^2 + 1)}{s^2 + 1} = \frac{s^3 + 6s^2 + 11s + 6}{s^2 + 1} \]

\[ \implies Y(s) = \frac{s^3 + 6s^2 + 11s + 6}{(s + 1)(s + 3)(s^2 + 1)} = \frac{A}{s + 1} + \frac{B}{s + 3} + \frac{Cs + D}{s^2 + 1}. \]

Expanding this out, we have

\[ s^3 + 6s^2 + 11s + 6 = A(s + 3)(s^2 + 1) + B(s + 1)(s^2 + 1) + (Cs + D)(s + 1)(s + 3). \]

It’s debatable which method is quicker for solving this method. I will choose to simplify by solving for the simple roots \( s = -1 \) and \( s = -3 \) first, then expand whatever is left over. We have

\[ s = -1 \implies 0 = A(-4) \implies A = 0 \]

and

\[ s = -3 \implies 0 = B(-20) \implies B = 0. \]
This simplifies our life significantly! We now have
\[ s^3 + 6s^2 + 11s + 6 = (Cs + D)(s^2 + 4s + 3) = Cs^3 + (4C + D)s^2 + (3C + 4D)s + 3D. \]

It follows that \( C = 1 \) and \( D = 2 \) (from the first and last coefficients) so that we have
\[
Y(s) = \frac{s + 2}{s^2 + 1} = \frac{s}{s^2 + 1} + \frac{2}{s^2 + 1}.
\]

It follows that the solution is
\[
y(x) = \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 1} \right\} + 2\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \right\} = \cos(x) + 2\sin(x).
\]

**Suggested Problems**

1. Use the Laplace Transform method to solve the following differential equations:

   (a) \[
   \begin{cases} 
   y' - y = e^x, \\
   y(0) = 1 
   \end{cases}
   \]

   (b) \[
   \begin{cases} 
   y' + 2y = 5\sin(x), \\
   y(0) = 1 
   \end{cases}
   \]

   (c) \[
   \begin{cases} 
   y'' + 6y' + 5y = 0, \\
   y(0) = 4, \\
   y'(0) = 0 
   \end{cases}
   \]

   (d) \[
   \begin{cases} 
   y'' - 7y' + 12y = 0, \\
   y(0) = 1, \\
   y'(0) = 1 
   \end{cases}
   \]

   (e) \[
   \begin{cases} 
   y'' - 4y' + 12y = e^{2x}, \\
   y(0) = 0, \\
   y'(0) = 0 
   \end{cases}
   \]

   (f) \[
   \begin{cases} 
   4y'' - 4y' + y = e^x, \\
   y(0) = -1, \\
   y'(0) = 0 
   \end{cases}
   \]

   (g) \[
   \begin{cases} 
   y'' + 2y' + 5y = 16xe^x, \\
   y(0) = 0, \\
   y'(0) = 0 
   \end{cases}
   \]

   (h) \[
   \begin{cases} 
   y'' - y' - y + y = 0, \\
   y(0) = 4, \\
   y'(0) = 0 
   \end{cases}
   \]

   (i) \[
   \begin{cases} 
   y'' - 4y'' + 5y = 0, \\
   y(0) = 5, \\
   y'(0) = 5, \\
   y''(0) = 0 
   \end{cases}
   \]