Section 1: Piecewise Functions

In the motivation for Laplace Transforms, we were led to believe that one of the primary advantages of this method is that it easily handles non-smooth and even discontinuous forcing functions. In order to handle such cases, we must expend a little energy developing the framework for the Laplace transform of piecewise-defined functions.

Consider computing the Laplace transform of the following function:

\[ f(x) = \begin{cases} 
  x, & 0 \leq x < 1 \\
  2 - x, & 1 \leq x < 2 \\
  0, & x \geq 2 
\end{cases} \]  

(1)

Such a function is commonly called a “tent” function.

This could model, for instance, an external signal that begins to climb at \( x = 0 \), then begins to fall at \( x = 1 \), and eventually reaches zero (i.e. no signal) at \( x = 2 \). Such forcing functions were notoriously difficult to handle in the classical differential equation setting since we essentially had to solve the differential equation independently in each region—i.e. we had to solve the differential equation three times!

Now consider computing the Laplace transform of such a function. From
the definition, we have that

\[
\mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x) \, dx
\]

\[
= \int_0^1 xe^{-sx} \, dx + \int_1^2 (2-x)e^{-sx} \, dx
\]

\[
= \left[-\frac{e^{-s}}{s} + \frac{1}{s} \int_0^1 e^{-sx} \, dx\right] - 2\frac{e^{-2s}}{s} + 2\frac{e^{-s}}{s}
\]

\[
+ \left[2\frac{e^{-2s}}{s} - \frac{e^{-s}}{s} - \frac{1}{s} \int_1^2 e^{-sx} \, dx\right]
\]

\[
= \frac{1}{s^2} (1 - 2e^{-s} + e^{-2s})
\]

where we have applied integration by parts several times and cleaned up the resulting expression.

That was a fair bit of work, but we should be relatively happy with the outcome. This tells us that, in the Laplace transform world, piecewise-defined functions correspond to a single function of \(s\). We will be able to solve differential equations with piecewise-defined forcing terms in exactly the same way as we have been traditional forcing terms.

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Section 2: Heaviside Functions

It is not convenient to apply the definition of a Laplace transform every time we use one. Rather, we want to develop a system of rules for handling piece-wise defined functions. The key to accomplishing this is the following.

Definition 1

The Heaviside function centered at \(c \geq 0\) is given by

\[
u_c(x) = \begin{cases} 
0, & 0 \leq x < c \\
1, & x \geq c
\end{cases}
\]

The Heaviside function can be thought of as an “on”/“off” switch with a trigger value \(c\). If we look to the left of \(c\), the function evaluates to zero (the “off” state), and if we look to the right of \(c\), the function evaluates to one (the “on” state).

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2
The importance of the Heaviside function lies in the fact that it can be combined with itself and other functions to generalize the notion of turning functions “on” or “off” over certain regions of $x$. In particular, for $d > c$ we can define

$$u_c(x) - u_d(x) = \begin{cases} 
0, & 0 \leq x < c \\
1, & c \leq x < d \\
0, & x \geq d
\end{cases}$$

In other words, we are only in the “on” state in the region $c \leq x < d$; otherwise, we are “off”. So this form allows us to define bounded intervals which are “on”.

Of course, what we are interested in turning “off” and “on” is not simply the value one. Rather, we are manipulating functions. In particular, consider the piecewise define function defined earlier:

$$f(x) = \begin{cases} 
x, & 0 \leq x < 1 \\
2 - x, & 1 \leq x < 2 \\
0, & x \geq 2
\end{cases}$$

What this definition means is that the function $f_1(x) = x$ is “on” in the region $0 \leq x < 1$, and then turned “off” at $x = 1$ when the new function $f_2(x) = 2 - x$ is turned “on”. Finally, at $x = 2$, $f_2(x) = 2 - x$ is turned “off” and the trivial function $f_3(x) = 0$ is turned “on”.

In fact, we can make use of exactly this intuition! For $f_1(x) = x$ to be turned “on” in the region $0 \leq x < 1$, we need to have

$$(1 - u_1(x))f_1(x) = (1 - u_1(x))x$$

where we notice that $1 - u_1(x)$ is “on” for $0 \leq x < 1$ and “off” for $x \geq 1$. Similarly, the idea of turning $f_2(x) = 2 - x$ “on” at $x = 1$ and “off” at $x = 2$ is captured by

$$(u_1(x) - u_2(x))f_2(x) = (u_1(x) - u_2(x))(2 - x).$$

Finally, we can turn $f_3(x) = 0$ “on” at $x = 2$ with

$$u_2(x)f_3(x) = 0.$$ 

It follows that the piecewise defined function can be written in terms of Heaviside functions as

$$f(x) = (1 - u_1(x))x + (u_1(x) - u_2(x))(2 - x)$$

$$= x + 2u_1(x)(1 - x) + u_2(x)(x - 2)$$
We have already seen that we could compute the Laplace transform of piece-wise defined functions, so let’s see how the Laplace transforms handles the Heaviside function. First of all, by the definition we can see that

\[ \mathcal{L}\{u_c(x)\} = \int_0^\infty u_c(x)e^{-sx}\,dx = \lim_{A\to\infty} \int_c^A e^{-sx} = \frac{e^{-cs}}{s}, \quad s > 0. \]

In particular, we notice that this generalizes for the case \( c = 0 \), corresponding to a function which is always “on”, to the identity

\[ \mathcal{L}\{u_0(x)\} = \mathcal{L}\{1\} = \frac{1}{s}. \]

Whenever we see a term \( e^{-cs} \) in the transformed world, therefore, we will immediately suspect that the Heaviside function is involved. Notice that we also have the inverse identity

\[ \mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\} = u_c(x). \]

We now want to consider what happens to functions which are turned “off” or “on” at a particular value. We know that we can formulate this intuition using Heaviside functions, so this is really a question of how we take Laplace transforms of functions which interact with Heaviside functions. We have the following result.

**Theorem 1**

Suppose \( F(s) = \mathcal{L}\{f(x)\} \) and \( u_c(x) \) is the Heaviside function centered at \( c \geq 0 \). Then

\[ \mathcal{L}\{u_c(x)f(x-c)\} = e^{-cs}F(s) \]

and, conversely,

\[ \mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(x)f(x-c). \]
Proof

By definition, we have

\[
\mathcal{L}\{u_c(x)f(x-c)\} = \int_0^\infty u_c(x)e^{-sx}f(x-c)\,dx
\]

\[
= \int_c^\infty e^{-sx}f(x-c)\,dx
\]

\[
= e^{-cs} \int_0^\infty e^{-s\tilde{x}}f(\tilde{x})\,d\tilde{x}
\]

\[
= e^{-cs} F(s)
\]

where we have made the substitution \(\tilde{x} = x - c\) (so that \(dx = d\tilde{x}\) and \(e^{-sx} = e^{-sc}e^{-s\tilde{x}}\)).

**Note:** The trick with applying this result will be to make sure that the function multiplying the Heaviside function is *always* arranged in factors of \(x - c\). Otherwise, the result does not apply and our answer will be wrong!

**Example 1**

Determine the Laplace transform of \(f(x) = x^2u_1(x)\).

**Solution:** We want to use our Theorem, but we cannot directly evaluate

\[
\mathcal{L}\{x^2u_1(x)\}
\]

because \(f(x) = x^2\) is not factored according to \(x - 1\). This can be corrected by adding and subtracting terms appropriately. In this case, we notice that we have

\[
(x - 1)^2 = x^2 - 2x + 1.
\]

Rearranging, we have

\[
x^2 = (x - 1)^2 + 2x - 1 = (x - 1)^2 + 2(x - 1) + 1
\]
where we had added and subtracted terms as appropriate. Finally, we have

\[ \mathcal{L} \{ x^2 u_1(x) \} = \mathcal{L} \{ ((x - 1)^2 + 2(x - 1) + 1)u_1(x) \} \]
\[ = e^{-s} \left( \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right). \]

Notice that in the last step we have ignored the time-shift! This is because, making the substitution \( \tilde{x} = x - 1 \), we have \( f(x - 1) = f(\tilde{x}) = \tilde{x}^2 + 2\tilde{x} + 1 \). This is the function corresponding to the Laplace transform \( F(s) \) in Theorem 1.

**Example 2**

Determine the inverse Laplace transform of

\[ F(s) = e^{-\pi s} \frac{4}{s^2 + 16}. \]

**Solution:** We have the strong indication from the \( e^{-\pi s} \) in the transform that there will be a Heaviside function \( u_\pi(x) \) in our solution. In particular, we expect the shift \( x - \pi \). First of all, however, we recognize that

\[ \mathcal{L}^{-1} \left\{ \frac{4}{s^2 + 16} \right\} = \sin(4x). \]

Applying our shift \( x - \pi \) to this form, we have that

\[ \mathcal{L}^{-1} \left\{ e^{-\pi s} \frac{4}{s^2 + 16} \right\} = u_\pi(x) \sin(4(x - \pi)). \]

**Example 3**

Use Theorem 1 to determine the Laplace transform of

\[ f(x) = \begin{cases} 
  x, & 0 \leq x < 1 \\
  2 - x, & 1 \leq x < 2 \\
  0, & x \geq 2
\end{cases}. \]
Solution: We know from our earlier work that \( f(x) \) can be written in the form

\[
f(x) = x + 2u_1(x)(1 - x) + u_2(x)(x - 2).
\]

In order to determine the Laplace transform, we need to compute

\[
\mathcal{L}\{f(x)\} = \mathcal{L}\{x\} + 2\mathcal{L}\{u_1(x)(1 - x)\} + \mathcal{L}\{u_2(x)(x - 2)\}.
\]

The only trick at this point is that we need each term multiplying a Heaviside function \( u_c(x) \) to be expressed in terms of the difference \( x - c \). In this case, we are almost done! We already have the differences \( x - 1 \) and \( x - 2 \) explicitly in the equations (this is not generally the case!). We may choose one final piece of simplification by get

\[
\mathcal{L}\{f(x)\} = \mathcal{L}\{x\} - 2\mathcal{L}\{u_1(x)(x - 1)\} + \mathcal{L}\{u_2(x)(x - 2)\}
\]

\[
= \frac{1}{s^2} (1 - 2e^{-s} + e^{-2s})
\]

as before.
Suggested Problems

1. Write the following piecewise-defined functions $f(x)$ as a single expression involving Heaviside functions:

   (a) $f(x) = \begin{cases} 
   1, & 0 \leq x < 1 \\
   -1, & 1 \leq x < 2 \\
   0, & x \geq 2.
   \end{cases}$

   (b) $f(x) = \begin{cases} 
   1 - x, & 0 \leq x < 1 \\
   x - 1, & 1 \leq x < 2 \\
   1, & x \geq 2.
   \end{cases}$

   (c) $f(x) = \begin{cases} 
   \sin(x), & 0 \leq x < \pi \\
   \cos(x), & \pi \leq x < 2\pi \\
   0, & x \geq 2\pi.
   \end{cases}$

   (d) $f(x) = \begin{cases} 
   0, & 0 \leq x < 2 \\
   -1, & 2 \leq x < 3 \\
   1, & x \geq 3.
   \end{cases}$

2. Determine the Laplace Transforms of the following piecewise-defined functions $f(x)$:

   (a) $f(x) = \begin{cases} 
   2, & 0 \leq x < 1 \\
   1, & 1 \leq x < 3 \\
   0, & x \geq 3.
   \end{cases}$

   (b) $f(x) = \begin{cases} 
   x, & 0 \leq x < 1 \\
   x^2 - 2x, & 1 \leq x < 2 \\
   0, & x \geq 2.
   \end{cases}$

3. Determine the Inverse Laplace Transforms of the following functions $F(s)$:

   (a) $F(s) = \frac{1}{s} \left( 1 + e^{-s} \right)$

   (b) $F(s) = \frac{1}{s} \left( 1 + \frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s^2} \right)$

   (c) $F(s) = \frac{e^{-s}}{s+1} + \frac{e^{-2s}}{s+2}$

   (d) $F(s) = \frac{s+1}{s^2 + 4s + 8} (1 + e^{\pi s})$