Math 133A, Fall 2015, Term Test II Study Sheet
Ordinary Differential Equations

Date: Thursday, November 19
Time: 1:30-2:45 p.m.
Lecture Section: 002
Instructor: Matthew Johnston

Surname (Family Name): ____________________________
Given Name: ____________________________
SJSU Student ID Number: ____________________________

Instructions

1. Fill out this cover page completely.

2. Answer questions in the space provided, using scratch paper for rough work.

3. Show all the work required to obtain your answers.

4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

FOR EXAMINERS’ USE ONLY

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1. Short Answer:
   (a) Convert the following higher-order differential equations in a system of first-order differential equations (all derivatives with respect to $x$):
   
   (i) $y'' + 5y' + 8y = \sin(x)$
   (ii) $y'' - 4xy = e^x$
   (iii) $(x + 1)y'' + xy' + x^2y = 1$
   (iv) $y^{(4)} + y'' = 0$

   (b) Determine the value of $c$ for which the following differential equations, corresponding to a mechanical pendulum/spring system obeying Newton’s second law, are critically damped:

   (i) $4x'' + cx' + x = 0$
   (ii) $9x'' + cx' + 4x = 0$
   (iii) $x'' + cx' + 3x = 0$
   (iv) $6x'' + cx' + \frac{5}{3}x = 0$

   (c) Supposing that a system of differential equations has the given general solution, determine the fundamental matrix $\Phi(t)$ with the property $\Phi(0) = I$.

   (i) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t}$
   (ii) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + C_2 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$
   (iii) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-2t} \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) \right)$
   + $C_2 e^{-2t} \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) \right)$

   2. Linear Systems and Vector Fields:
   For the initial value problems, sketch the vector field and then find the solution. Clearly label the nullclines and the identify the direction of flow ($↗$, $↘$, $↖$, $↙$) in each region.

   (a) \begin{align*}
   x' &= -x + 2y \\
   y' &= -x - 4y \\
   x(0) &= 3, \ y(0) = -3
   \end{align*}
   
   (b) \begin{align*}
   x' &= x - 2y \\
   y' &= -x + 2y \\
   x(0) &= 0, \ y(0) = 3
   \end{align*}
   
   (c) \begin{align*}
   x' &= -3x - 9y \\
   y' &= x + 3y \\
   x(0) &= 1, \ y(0) = 0
   \end{align*}
   
   (d) \begin{align*}
   x' &= -x + 2y \\
   y' &= -x + y \\
   x(0) &= 2, \ y(0) = -1
   \end{align*}
   
   (e) \begin{align*}
   x' &= \sqrt{2}x + 3y \\
   y' &= -3x + \sqrt{2}y \\
   x(0) &= \sqrt{2}, \ y(0) = \sqrt{2}
   \end{align*}
   
   (f) \begin{align*}
   x' &= -x + y \\
   y' &= -4x - 5y \\
   x(0) &= 0, \ y(0) = -2
   \end{align*}
3. Second-Order Differential Equations:

(a) Determine the trial function $y_p(x)$ for the following second-order differential equations. Do not attempt to solve for the constants.

(i) $y'' + 2y' - 3y = e^{-x} + e^x$
(ii) $y'' + 2y' - 3y = xe^{-x}$
(iii) $y'' + 2y' - 3y = 2xe^{-3x}$
(iv) $y'' + 4y' + 5y = e^{-2x}$
(v) $y'' + 4y' + 5y = \sin(x) + \cos(x)$
(vi) $y'' + 4y' + 5y = xe^{-2x} \sin(x)$

(b) Solve the following initial value problems:

(i) \[
\begin{align*}
2y'' + 5y' - 3y &= 10 \sin(x), \\
y(0) &= 1, \\
y'(0) &= 0
\end{align*}
\]
(ii) \[
\begin{align*}
y'' - 2y' + 5y &= 5x - 2, \\
y(0) &= 1, \\
y'(0) &= -2
\end{align*}
\]
(iii) \[
\begin{align*}
y'' + 3y' + 2y &= 6e^x, \\
y(0) &= 1, \\
y'(0) &= 0
\end{align*}
\]
(iv) \[
\begin{align*}
y'' + 9y &= 12 \sin \left( \frac{3}{2}x \right), \\
y(0) &= 0, \\
y'(0) &= 0
\end{align*}
\]

4. Applications: (Mechanical Systems)

(a) A 1 kg mass is attached to the end of a pendulum which has a restoring constant of 5 N/m and damping constant of 4 N/(m/s). Suppose the system is driven by an external force of $f(t) = 8 \sin(t)$ N. Formulate a second-order differential equation to describe the system, and solve it. Rewrite the forcing function and periodic portion of the solution $x(t)$ in the form $R \cos(\omega_0 t + \delta)$. Comment on the long-term behavior of the solution, whether it depends upon the initial conditions, and the relationship between the forcing and long-term response.

(b) An 8 kg mass is attached to the end of a spring which has a restoring constant of 1 N/m and damping constant of 6 N/(m/s). Suppose the system is driven by an external force of $f(t) = (5 \sin(\frac{1}{2}t) + 5 \cos(\frac{1}{2}t))$ N. Formulate a second-order differential equation to describe the system, and solve it. Rewrite the forcing function and periodic portion of the solution $x(t)$ in the form $R \cos(\omega_0 t + \delta)$. Comment on the long-term behavior of the solution, whether it depends upon the initial conditions, and the relationship between the forcing and long-term response.

5. Applications: (Chemical Reactions)

Consider the reversible chemical reaction $X \rightleftharpoons Y$ occurring in a tank subject to continuous inflow of $X$ and outflow of both $X$ and $Y$. Suppose the process can be modeled by the following system of first-order differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= -5x + 2y + 7 \\
\frac{dy}{dt} &= 4x - 3y
\end{align*}
\]

(a) Solve the non-homogeneous system of differential equations (1) for $x(t)$ and $y(t)$.

(b) Describe the long-term behavior of the chemical species $X(t)$ and $Y(t)$. 
THIS PAGE IS FOR ROUGH WORK