Instructions

1. Fill out this cover page completely.

2. Answer questions in the space provided, using scratch paper for rough work.

3. Show all the work required to obtain your answers.

4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

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1. Short Answer:

(a) Convert the following higher-order differential equation into a first-order system of differential equations (all derivatives with respect to $x$): $x^2y''+xy'-y = \sin(x)$.

(b) Suppose a differential equation has the general solution:

$$
\begin{bmatrix}
  x(t) \\
  y(t)
\end{bmatrix}
= C_1 \begin{bmatrix}
  1 \\
  2
\end{bmatrix} e^t + C_2 \left( \begin{bmatrix}
  1 \\
  2 
\end{bmatrix} t + \begin{bmatrix}
  1 \\
  0
\end{bmatrix} \right) e^t.
$$

Determine the fundamental matrix $\Phi(t)$ with the property $\Phi(0) = I$.

2. True/False:

(a) Suppose $A \in \mathbb{R}^{2\times2}$ and $B \in \mathbb{R}^{2\times2}$ (two by two matrices). Then it must be true that $AB = BA$.

[True / False]

(b) The matrix $A = \begin{bmatrix}
  -2 & 6 \\
  3 & -9
\end{bmatrix}$ has an eigenvalue of zero.

[True / False]

(c) The function $y_p(x) = \sin(x)$ is a particular solution of $y'' + y' = \sin(x) + \cos(x)$.

[True / False]
3. **Linear Systems and Vector Fields:**

Consider the following differential equation:

\[
\begin{align*}
  x' &= x - 2y \\
  y' &= x - y
\end{align*}
\]  

(a) Sketch the vector field in the \((x, y)\)-plane. Clearly label the nullclines and the identify the direction of flow (↗, ↘, ↖, ↙) in each region.

(b) Determine the general solution \(\{x(t), y(t)\}\) of the \((1)\).
4. Second-Order Differential Equations:

Consider the following second-order differential equation:

\[ y'' + 2y' + y = g(x). \] \hspace{1cm} (2)

(a) Determine the complementary solution \( y_c(x) \) of (2).

(b) Set-up the trial function \( y_p(x) \) for the following \( g(x) \) (do not attempt to evaluate the constants!):

(i) \( g(x) = x \sin(2x) \)

(ii) \( g(x) = e^{-x} \)

(c) Determine the particular solution \( y_p(x) \) for (2) if \( g(x) = x^2 - 6 \). (Now you have to solve the constants!)
5. Applications: (Mechanical Systems)

A 2 kg mass is attached to the end of a pendulum which has a restoring constant of 10 N/m and damping constant of 4 N/(m/s). Suppose the system is driven by an external force of $f(t) = (4 \cos(t) + 8 \sin(t))$ N.

(a) Set up a second-order differential equation which models the motion of the pendulum as a function of time, $x(t)$.

(b) Determine the general solution $x(t)$ and then briefly describe the pendulum’s behavior over time.
6. Applications: (Chemical Reactions)

Consider the reversible chemical reaction \( X \rightleftharpoons Y \) occurring in a tank subject to continuous inflow of \( X \) and outflow of both \( X \) and \( Y \). Suppose the process can be modeled by the following system of first-order differential equations:

\[
\begin{align*}
    x' &= -3x + 2y + 10 \\
    y' &= x - 4y.
\end{align*}
\]  

(3)

(a) Given that the matrix \( A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \) has eigenvalue/eigenvector pairs \( \lambda_1 = -2 \), \( \mathbf{v}_1 = (2, 1) \), and \( \lambda_2 = -5 \), \( \mathbf{v}_2 = (-1, 1) \), determine the general solution of (3).

**Hint:** Do not forget to account for the additional +10 term!