3. Solve the following first-order differential equations:

(a) \( y' = xy + y, \ y(0) = 1 \)

General solution: \( y(x) = Ce^{\frac{x^2}{2}} + x \)
Particular solution: \( y(x) = e^{\frac{x^2}{2}} + x \)

(b) \( x^2y' + (x + 1)y = \frac{1}{x}, \ y(1) = 0 \)

General solution: \( y(x) = \frac{1 + Ce^{\frac{1}{x}}}{x} \)
Particular solution: \( y(x) = \frac{1 - e^{\frac{1}{x}} - 1}{x} \)

(c) \( 12yy' + 8x^2 = \frac{x}{y}, \ y(0) = 1/2 \)

General solution: \( y(x) = \frac{1}{2} \left( 1 + Ce^{-x^2} \right)^{\frac{1}{2}} \)
Particular solution: \( y(x) = \frac{1}{2} \)

(d) \( 2x^2y + (xy^2 - x^3)y' = 0, \ y(0) = -1/2 \)

General solution: \( y(x) = \frac{1 \pm \sqrt{1 - C^2x^2}}{C} \)
Particular solution: \( y(x) = -\frac{1 + \sqrt{1 - 16x^2}}{4} \)

(e) \((y^2 - 1)dx + 2xy dy = 0, \ y(2) = 0 \)

General solution: \( x(y^2 - 1) = C \)
Particular solution: \( x(y^2 - 1) = -2 \)

(f) \( dx + x(ln(y) + 1)dy = 0, \ y(1) = 2 \)

General solution: \( \ln(x) + y\ln(y) = C \)
Particular solution: \( \ln(x) + y\ln(y) = \ln(4) \)
5. Suppose the temperature (in Celsius) of a lake equilibrates with the seasonal temperature according to the equation

\[ \frac{dT}{dt} = \cos\left(\frac{\pi}{6}t\right) - \frac{\pi}{6}T \]  

where the time \( t \) is in months. (Note that fluctuations complete a full period every 12 months).

(a) Find the general solution of (1).

**Solution:** \( T(t) = Ce^{-\frac{\pi}{6}t} + \frac{3}{\pi} \left( \cos\left(\frac{\pi}{6}t\right) + \sin\left(\frac{\pi}{6}t\right) \right) \)

(b) Write the solution \( T(t) \) in the form \( T(t) = T_{tr}(t) + T_{sp}(t) \) where \( T_{tr}(t) \) is the transient portion of the solution (decays to zero as \( t \to \infty \)) and \( T_{sp}(t) \) is the steady state periodic portion of the solution.

**Solution:** \( T_{tr}(t) = Ce^{-\frac{\pi}{6}t} \), \( T_{sp}(t) = \frac{3}{\pi} \left( \cos\left(\frac{\pi}{6}t\right) + \sin\left(\frac{\pi}{6}t\right) \right) \)

(c) Write the steady state periodic solution \( T_{sp}(t) \) in the form \( T_{sp}(t) = A \cos(\omega t - \alpha) \).

**Solution:** \( T_{sp}(t) = \frac{3\sqrt{2}}{\pi} \cos\left(\frac{\pi}{6}t - \frac{\pi}{4}\right) \)

(d) How does the amplitude \( A \) compare with the amplitude of the forcing term (i.e. the season variation)?
Solution: Forcing amplitude: $A = 1$; Steady state amplitude: $A = 3\sqrt{2}/\pi \approx 1.3505$. (Note the real seasonal variation is $6/\pi$, however.)

(e) The phase shift $\alpha$ represents the lag in the lake’s response to the seasonal variable in temperature (e.g. the lake will not freeze as soon as temperatures dip below freezing and, similarly, the lake will not thaw at the first hint of spring; rather they will take some time). How long (in months) is the lag between the season variation and the lake’s response at steady state?

Solution: We can write $(\pi/6)t - \pi/4 = (\pi/6)(t - 3/2)$ so that the phase shift is $3/2 = 1.5$. That is to say, the temperature of the lake lags behind the seasonal variation by one and a half months.

6. Convert the following differential equations into a system of first-order differential equations:

(a) $9x'' + x' + 5x = e^t$

Solution: $x'_1 = x_2$, $x'_2 = -(5/9)x_1 - (1/9)x_2 + (1/9)e^t$

(b) $x^{(n)} - x = 0$

Solution: $x'_i = x_{i+1}$ for $i = 1, \ldots, n-1$, and $x'_n = x_1$

(c) $x'' \cdot x' + x' \cdot x = 1$

Solution: $x'_1 = x_2$, $x'_2 = -x_1 + \frac{1}{x_2}$

7. Solve the following linear systems of differential equations and sketch the vector field diagram in the $(x,y)$-plane:

(a) \[
\begin{cases}
    x' = x - 8y \\
    y' = -x + 3y
\end{cases}
\]

(b) \[
\begin{cases}
    x' = 2x - 4y \\
    y' = x - 2y
\end{cases}
\]

(c) \[
\begin{cases}
    x' = -12x - 18y \\
    y' = 5x - 6y
\end{cases}
\]

$x(0) = 0$, $y(0) = 15$

Solution (a):

$x(t) = C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{5t}$

Solution (b):

$x(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$
Solution (c):

\[
x(t) = 3e^{-9t} \left( \begin{bmatrix} -3 \\ 5 \\ \end{bmatrix} \cos(9t) - \begin{bmatrix} 9 \\ 0 \\ \end{bmatrix} \sin(9t) \right) \\
+ (1)e^{-9t} \left( \begin{bmatrix} -3 \\ 5 \\ \end{bmatrix} \sin(9t) + \begin{bmatrix} 9 \\ 0 \\ \end{bmatrix} \cos(9t) \right) \\
= e^{-9t} \left( \begin{bmatrix} 0 \\ 15 \\ \end{bmatrix} \cos(9t) + \begin{bmatrix} -30 \\ 5 \\ \end{bmatrix} \sin(9t) \right).
\]

8. Solve the following nonhomogeneous linear systems of differential equations:

\[
(a) \begin{cases} 
    x' = 3x + 2y - 25t \\
    y' = -x + y 
\end{cases}
\]

Solution:

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{2t} \sin(t) + \cos(t) \\ -e^{2t} \sin(t) \end{bmatrix} C_1 + \begin{bmatrix} 2e^{2t} \sin(t) \\ e^{2t} (\cos(t) - \sin(t)) \end{bmatrix} C_2 + \begin{bmatrix} 5t - 1 \\ 5t + 4 \end{bmatrix}
\]

\[
(b) \begin{cases} 
    x' = 6x + 4y + e^{-t} \\
    y' = -14x - 9y - e^{-t} 
\end{cases}
\]

Solution:

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (4t + 1)e^{-t} - e^{-2t} \\ -(7t + 1)e^{-t} - 2e^{-2t} \end{bmatrix}
\]

9. Consider the reversible chemical reaction \( X \rightleftharpoons Y \). Denoting the concentrations of the chemicals as \( x = [X] \) and \( y = [Y] \) we can write the system of governing differential equations as

\[
\begin{cases} 
    x' = -\alpha x + \beta y, \quad x(0) = x_0 \\
    y' = \alpha x - \beta y, \quad y(0) = y_0 
\end{cases}
\]

where \( \alpha, \beta > 0 \) are the rates of the forward and backward reactions, respectively.

(a) Solve this differential equation for the values \( \alpha = 4, \beta = 1, x_0 = 4 \) and \( y_0 = 0 \). What is the long-term behavior of the system? Does this make sense in terms of the physical set-up?

Solution:

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{4}{16} (4e^{-5t} + 1) \\ \frac{1}{16} (1 - e^{-5t}) \end{bmatrix}
\]
Long-term behavior: As $t \to \infty$, $x(t) \to 4/5$ and $y(t) \to 16/5$. System settles to balanced concentration levels. Notice that $x(t) + y(t) = x(0) + y(0) = 4$. That is to say, the amount of $X$ and $Y$ is conserved. This makes sense in terms of the physical set-up.

(b) Now solve the differential equation for the general values $\alpha, \beta, x_0$ and $y_0$. Show that the long-term behavior is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (x_0 + y_0).$$

Solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{\alpha + \beta} \begin{bmatrix} \beta (x_0 + y_0) + (\alpha x_0 - \beta y_0) e^{-(\alpha+\beta)t} \\ \alpha (x_0 + y_0) - (\alpha x_0 - \beta y_0) e^{-(\alpha+\beta)t} \end{bmatrix}$$

Stated result clearly holds in limit as $t \to \infty$.

(c) Suppose that there is continuous fixed inflow of $X$ and continuous fixed outflow of $Y$. Assuming the parameter values in (a) and appropriate values for the inflow/outflow, we can model this by

$$\begin{cases} x' = -4x + y + 1, & x(0) = 4 \\ y' = 4x - y - 1, & y(0) = 0 \end{cases}$$

Solve this initial value problem. What is the long-term behavior of the system? How does it differ from the solution of (a)?

Solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 + 3e^{-5t} \\ 3 - 3e^{-5t} \end{bmatrix}$$

Long-term behavior: As $t \to \infty$, $x(t) \to 1$ and $y(t) \to 3$. System settles to balanced concentration levels which are (slightly) different than the system with no inflow/outflow.

10. Consider a 9 kg mass on a spring. Suppose that it takes a force of 1 Newton to displace the mass 1 meter. Allowing friction and forcing to be present, but not specified, this gives the model

$$9x'' + cx' + x = g(t). \tag{2}$$

(a) Determine the value of $c \in \mathbb{R}$ for which (2) is critically damped.

Solution: $c = 6$
(b) Solve (2) for the values $c = 0$ and $g(t) = 2\cos(1/2)t$ and the initial conditions $x(0) = x'(0) = 0$. What is the maximum amplitude of the solution? How does this compare with the forcing amplitude of two?

Solution: $x(t) = \frac{8}{5} \left( \cos \left( \frac{1}{3} t \right) - \cos \left( \frac{1}{2} t \right) \right) = \frac{16}{5} \sin \left( \frac{5}{12} t \right) \sin \left( -\frac{1}{12} t \right)$, so amplitude is $A = 16/5$, which is larger than the forcing amplitude of 2 (i.e. there is resonance).

(c) Solve (2) for the values $c = 0$ and $f(t) = 2\cos((1/3)t)$ and the initial conditions $x(0) = x'(0) = 0$. What is the maximum amplitude of the solution? How does this compare with the forcing amplitude of two?

Solution: $x(t) = \frac{t}{3} \sin \left( \frac{1}{3} t \right)$, max amplitude is infinity

(d) Solve (2) for the values $c = 3$ and $f(x) = 2\cos((1/3)t)$ and the initial conditions $x(0) = x'(0) = 0$. What is the maximum amplitude of the solution? How does this compare with the forcing amplitude of two?

Solution: $x(t) = -\frac{4\sqrt{3}}{3} e^{-\frac{1}{3}t} \sin \left( \frac{\sqrt{3}}{6} t \right) + 2 \sin \left( \frac{1}{3} t \right)$, maximum amplitude of steady state solution is 2, same as forcing amplitude (no resonance).

11. Evaluate the following:

(a) $\mathcal{L}^{-1} \left\{ \frac{2s + 3}{s^2 + 2s + 1} \right\}$

Solution: $(x + 2)e^{-x}$

(b) $\mathcal{L} \{ (t^2 - 2t)u_2(t) \}$

Solution: $\left( \frac{2}{s^3} \right) e^{-2s}$ (Note: $t^2 - 2t = (t - 2)^2 + 2(t - 2)$)

(c) $\mathcal{L}^{-1} \left\{ \left( \frac{2s - 2}{s(s^2 - 2s + 2)} \right) e^{-s} \right\}$

Solution: $(e^{t-1} \cos(t - 1) + e^{t-1} \sin(t - 1) - 1)u_1(t)$

12. Use the Laplace transform method to solve the following initial value problems:

(a) $x'' + x = f(t), \quad x(0) = 0, x'(0) = 0$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$$
Solution: \( x(t) = 1 - \cos(t) - (1 - \cos(t - 1))u_1(t) \) or, alternatively,

\[
x(t) = \begin{cases} 
1 - \cos(t), & 0 \leq t < 1 \\
\cos(t - 1) - \cos(t), & t \geq 1.
\end{cases}
\]

(b) \( x'' + 2x' + x = f(t), \quad x(0) = 0, x'(0) = 1 \) where

\[
f(t) = \begin{cases} 
0, & 0 \leq t < c \\
1, & c \leq t < d \\
0, & t \geq d
\end{cases}
\]

Solution: \( x(t) = te^{-t} + (1 - e^{-(t-c)} - (t-c)e^{-(t-c)})u_c(t) - (1 - e^{-t-d} - (t-d)e^{-t-d})u_d(t) \)

(c) \( x'' + 5x' + 4x = 3\delta(t - 1), \quad x(0) = 3, \quad x'(0) = 0 \)

Solution: \( x(t) = 4e^{-t} - e^{-4t} + u_1(t) (e^{-t-1} - e^{-4(t-1)}) \)