MATH 133A, Fall 2015, Final Exam
Supplemental Questions

Note that the final exam will be cumulative with a slight weighting toward the material since the second midterm (that is, Laplace Transforms). As with the term tests, no calculators will be permitted, but you will be permitted a one page cheat sheet.

Supplemental study material:

1. Material from or related to Assignments #1-11
2. Statements of Definitions and Results from class (proofs not required unless otherwise stated).
3. Solve the following first-order differential equations:
   (a) \( y' = xy + y, \quad y(0) = 1 \)
   (b) \( x^2y' + (x + 1)y = \frac{1}{x}, \quad y(1) = 0 \)
   (c) \( 12yy' + 8xy^2 = \frac{x}{y}, \quad y(0) = 1/2 \)
   (d) \( 2x^2y + (xy^2 - x^3)y' = 0, \quad y(0) = -1/2 \)
   (e) \( (y^2 - 1)dx + 2xy dy = 0, \quad y(2) = 0 \)
   (f) \( dx + x(\ln(y) + 1)dy = 0, \quad y(1) = 2 \)
4. Draw the slope fields for the following first-order differential equations (derivatives with respect to \( x \)):
   (a) \( y' = y^2 - 4 \)
   (b) \( y' = 2x - y + 1 \)
   (c) \( y' = \sqrt{1 - x^2 - y^2}, \quad x^2 + y^2 \leq 1 \)
5. Suppose the temperature (in Celsius) of a lake equilibrates with the seasonal temperature according to the equation

\[
\frac{dT}{dt} = \cos \left( \frac{\pi}{6} t \right) - \frac{\pi}{6} T
\] (1)

where the time \( t \) is in months. (Note that fluctuations complete a full period every 12 months).
(a) Find the general solution of (1).

(b) Write the solution $T(t)$ in the form $T(t) = T_{tr}(t) + T_{sp}(t)$ where $T_{tr}(t)$ is the transient portion of the solution (decays to zero as $t \to \infty$) and $T_{sp}(t)$ is the steady state periodic portion of the solution.

(c) Write the steady state periodic solution $T_{sp}(t)$ in the form $T_{sp}(t) = A \cos(\omega t - \alpha)$.

(d) How does the amplitude $A$ compare with the amplitude of the forcing term (i.e. the season variation)?

(e) The phase shift $\alpha$ represents the lag in the lake’s response to the seasonal variable in temperature (e.g. the lake will not freeze as soon as temperatures dip below freezing and, similarly, the lake will not thaw at the first hint of spring; rather they will take some time). How long (in months) is the lag between the season variation and the lake’s response at steady state?

6. Convert the following differential equations into a system of first-order differential equations:

(a) $9x'' + x' + 5x = e^t$

(b) $x^{(n)} - x = 0$

(c) $x'' \cdot x' + x' \cdot x = 1$

7. Solve the following linear systems of differential equations and sketch the vector field diagram in the $(x,y)$-plane:

(a) \[
\begin{align*}
    x' &= x - 8y \\
    y' &= -x + 3y
\end{align*}
\]

(b) \[
\begin{align*}
    x' &= 2x - 4y \\
    y' &= x - 2y
\end{align*}
\]

(c) \[
\begin{align*}
    x' &= -12x - 18y, & x(0) = 0 \\
    y' &= 5x - 6y, & y(0) = 15
\end{align*}
\]

8. Solve the following nonhomogeneous linear systems of differential equations:

(a) \[
\begin{align*}
    x' &= 3x + 2y - 25t \\
    y' &= -x + y
\end{align*}
\]

(b) \[
\begin{align*}
    x' &= 6x + 4y + e^{-t}, & x(0) = 0 \\
    y' &= -14x - 9y - e^{-t}, & y(0) = 1
\end{align*}
\]
9. Consider the reversible chemical reaction \( X \rightleftharpoons Y \). Denoting the concentrations of the chemicals as \( x = [X] \) and \( y = [Y] \) we can write the system of governing differential equations as

\[
\begin{align*}
x' &= -\alpha x + \beta y, \quad x(0) = x_0 \\
y' &= \alpha x - \beta y, \quad y(0) = y_0
\end{align*}
\]

where \( \alpha, \beta > 0 \) are the rates of the forward and backward reactions, respectively.

(a) Solve this differential equation for the values \( \alpha = 4, \beta = 1, x_0 = 4 \) and \( y_0 = 0 \). What is the long-term behavior of the system? Does this make sense in terms of the physical set-up?

(b) Now solve the differential equation for the general values \( \alpha, \beta, x_0 \) and \( y_0 \). Show that the long-term behavior is given by

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \rightarrow \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \frac{\alpha + \beta}{\alpha + \beta} (x_0 + y_0).
\]

(c) Suppose that there is continuous fixed inflow of \( X \) and continuous fixed outflow of \( Y \). Assuming the parameter values in (a) and appropriate values for the inflow/outflow, we can model this by

\[
\begin{align*}
x' &= -4x + y + 1, \quad x(0) = 4 \\
y' &= 4x - y - 1, \quad y(0) = 0
\end{align*}
\]

Solve this initial value problem. What is the long-term behavior of the system? How does it differ from the solution of (a)?

10. Consider a 9 kg mass on a spring. Suppose that it takes a force of 1 Newton to displace the mass 1 meter. Allowing friction and forcing to be present, but not specified, this gives the model

\[
9x'' + cx' + x = g(t).
\]

(a) Determine the value of \( c \in \mathbb{R} \) for which (2) is critically damped.

(b) Solve (2) for the values \( c = 0 \) and \( g(t) = 2 \cos((1/2)t) \) and the initial conditions \( x(0) = x'(0) = 0 \). What is the maximum amplitude of the solution? How does this compare with the forcing amplitude of two?
(c) Solve (2) for the values $c = 0$ and $f(t) = 2\cos((1/3)t)$ and the initial conditions $x(0) = x'(0) = 0$. What is the maximum amplitude of the solution? How does this compare with the forcing amplitude of two?

(d) Solve (2) for the values $c = 3$ and $f(x) = 2\cos((1/3)t)$ and the initial conditions $x(0) = x'(0) = 0$. What is the maximum amplitude of the solution? How does this compare with the forcing amplitude of two?

11. Evaluate the following:

(a) $\mathcal{L}^{-1}\left\{\frac{2s + 3}{s^2 + 2s + 1}\right\}$

(b) $\mathcal{L}\{ (t^2 - 2t)u_2(t) \}$

(c) $\mathcal{L}^{-1}\left\{ \left( \frac{2s - 2}{s(s^2 - 2s + 2)} \right) e^{-s} \right\}$

12. Use the Laplace transform method to solve the following initial value problems:

(a) $x'' + x = f(t), \quad x(0) = 0, \quad x'(0) = 0 \quad$ where

\[ f(t) = \begin{cases} 
1, & 0 \leq t < 1 \\
0, & t \geq 1
\end{cases} \]

(b) $x'' + 2x' + x = f(t), \quad x(0) = 0, \quad x'(0) = 1 \quad$ where

\[ f(t) = \begin{cases} 
0, & 0 \leq t < c \\
1, & c \leq t < d \\
0, & t \geq d
\end{cases} \]

(c) $x'' + 5x' + 4x = 3\delta(t - 1), \quad x(0) = 3, \quad x'(0) = 0.$