Math 32, Fall 2015, Term Test III
Calculus III

Date: Tuesday, November 24
Time: 10:30-11:45 a.m.
Lecture Section: 001
Instructor: Matthew Johnston

Last Name: 
First Name: 
SJSU Student ID Number: 

Instructions
1. Fill out this cover page completely.
2. Answer questions in the space provided, using scratch paper for rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

FOR EXAMINERS' USE ONLY

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1. Short Answer:

(a) From the contour plot of \( f(x, y) \) below, classify the critical point \((0, 0)\):

![Saddle point diagram]

(b) State conditions which are sufficient to guarantee that a critical point \((x^*, y^*)\) of \( f(x, y) \) is a local maximum. [Note: If you introduce place-holder terms like \( A, B, C, \) etc., make sure you define them!]

\[
A = \frac{\partial^2 f}{\partial x^2}(x^*, y^*) \quad \{ \begin{array}{c} AC - B^2 > 0 \\ A < 0 \end{array} \]

\[
B = \frac{\partial^2 f}{\partial x \partial y}(x^*, y^*)
\]

\[
C = \frac{\partial^2 f}{\partial y^2}(x^*, y^*)
\]

2. True/False:

(a) The point \((1, -1)\) is a critical point of \( f(x, y) = xy + x - y \).

True / False

\[
\begin{align*}
\frac{\partial f}{\partial x} &= y + 1 = 0 \quad \Rightarrow \quad y = -1 \\
\frac{\partial f}{\partial y} &= x - 1 = 0 \quad \Rightarrow \quad x = 1
\end{align*}
\]

(b) Every critical point \((x, y)\) of a multivariate function \( f(x, y) \) corresponds to either a local maxima or a local minima.

True / False

(c) The double integral corresponding to the volume below \( f(x, y) \geq 0 \) over the region with \( a \leq x \leq b \) and \( c \leq y \leq d \) is given by \( \int_c^d \int_a^b f(x, y) \, dx \, dy \).

True / False

(d) So long as \( f(x, y) \) is continuous on the involved region, we have that

\[
\int_0^{\sqrt{1-y}} \int_0^{1-x^2} f(x, y) \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-y}} f(x, y) \, dx \, dy.
\]

True / False

\[
\begin{align*}
y &= 1 - x^2 \\
\Rightarrow \quad 1 - y &= x^2 \\
\Rightarrow \quad x &= \sqrt{1-y}
\end{align*}
\]
3. Maxima/Minima:

Consider the following function:

\[ f(x, y) = x^2 y + 2xy - y^2 \]  \hspace{1cm} (1)

(a) Determine the critical points of \( f(x, y) \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2xy + 2y = 0 \Rightarrow 2y(x + 1) \\
\frac{\partial f}{\partial y} &= x^2 + 2x - 2y = 0
\end{align*}
\]

\[ y = 0 \quad \text{or} \quad x = -1 \]

\[ \Downarrow \]

\[ \begin{align*}
x^2 + 2x &= 0 \\
1 - 2 - 2y &= 0
\end{align*} \]

\[ \Rightarrow x(x + 2) = 0 \quad \Rightarrow \quad 2y = -1 \]

\[ \Rightarrow x = 0 \quad \text{or} \quad x = -2 \]

\[ \Rightarrow (0, 0), (-2, 1/2), (-1, -1/2) \]

(b) Classify the critical points found in (a) as either a local maximum, local minimum, or saddle point.

\[
\begin{align*}
A &= f_{xx}(x, y) = 2y \\
B &= f_{xy}(x, y) = 2x + 2 \\
C &= f_{yy}(x, y) = -2
\end{align*}
\]

\[ A + (0, 0) \Rightarrow AC - B^2 = (0)(-2) - (2)^2 = -4 < 0 \Rightarrow \text{saddle} !\]

\[ A + (-2, 0) \Rightarrow AC - B^2 = (0)(-2) - (-2)^2 = -4 < 0 \Rightarrow \text{saddle} !\]

\[ A + (-1, -1/2) \Rightarrow AC - B^2 = (-1)(-2) - (0)^2 = 2 < 0 \]

\[ A = -1 < 0 \quad \Rightarrow \text{max} !\]
4. Lagrange Multipliers:

Use Lagrange Multipliers to determine the global maximum and global minimum of

\[ f(x, y) = 2x + y^2 \]

restricted to the set

\[ x^2 + y^2 = 2. \]

Identify both the values of the maximum and minimum and the points where they occur.

(Hint: Remember to write the constraint in the standard form \( g(x, y) = 0 \)).

Also note \( 3 > 2 \sqrt{2} \).

\[
\mathcal{L}(x, y, \lambda) = 2x + y^2 + \lambda(x^2 + y^2 - 2)
\]

(1) \[ \mathcal{L}_x = 2 + 2\lambda x = 0 \]

(2) \[ \mathcal{L}_y = 2y + 2\lambda y = 0 \]

(3) \[ \mathcal{L}_\lambda = x^2 + y^2 - 2 = 0 \]

\[
\begin{align*}
\text{Also note:} & \quad 3 > 2 \sqrt{2} \\
\text{(1) } & \quad \lambda = -1 \\
\text{(2) } & \quad 2y(1 + \lambda) = 0 \\
\text{(3) } & \quad x^2 - 2 = 0 \\
\text{(4) } & \quad y = 0 	ext{ or } y = \pm 1
\end{align*}
\]

Points: \((\pm \sqrt{2}, 0), (1, \pm 1)\)

Check:

\[
\begin{align*}
f(\sqrt{2}, 0) &= 2\sqrt{2} \\
f(-\sqrt{2}, 0) &= -2\sqrt{2} \\
f(1, \pm 1) &= 2 + 1 = 3
\end{align*}
\]
5. Double Integration:

(a) Set up a double integral for computing the volume below a function \( f(x, y) \geq 0 \) above the following regions \( R \):

(i) \( R = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \pi, -3 \leq y \leq 2 \} \)

\[
\int_0^\pi \int_{-3}^2 f(x, y) \, dy \, dx = \int_0^2 \int_{-\sqrt{4-y^2}}^\sqrt{4-y^2} f(x, y) \, dx \, dy
\]

(ii) \( R = \{ (x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, 0 \leq y \leq 4 - x^2 \} \)

\[
\int_{-2}^2 \int_0^{4-x^2} f(x, y) \, dy \, dx = \int_0^4 \int_{\sqrt{4-y}}^2 f(x, y) \, dx \, dy
\]

(iii) \( R \) is the triangle with vertices (0,0), (3,3), and (-3,-3).

\[
\int_{-3}^3 \int_{-x}^3 f(x, y) \, dy \, dx = \int_0^3 \int_{-\sqrt{3-y}}^{\sqrt{3-y}} f(x, y) \, dx \, dy
\]

(b) Exchange the order of integration in the following double integrals:

(i) \( \int_0^2 \int_0^{1-\frac{1}{2}x} f(x, y) \, dy \, dx \)

\[
= \int_0^1 \int_0^{2-2y} f(x, y) \, dx \, dy
\]

(ii) \( \int_{-1}^1 \int_{\frac{y^2}{1-x}} f(x, y) \, dx \, dy \)

\[
= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x, y) \, dy \, dx
\]
(c) Evaluate the following double integrals.

(i) \[ \int_0^2 \int_0^1 \frac{2xy}{1+y^2} \, dy \, dx \]

\[ = \int_0^2 \left[ \int_0^1 \frac{2xy}{1+y^2} \, dy \right] \, dx \]

\[ = \int_0^2 \left[ \left. \frac{1}{2} \ln(1+y^2) \right|_{y=0}^{y=2} \right] \, dx \]

\[ = \int_0^2 \left[ \ln(1+4) - \ln(1) \right] \, dx \]

\[ = \ln(5) \int_0^2 \, dx \]

\[ = \ln(5) \left[ \frac{x^2}{2} \right]_{x=0}^{x=2} \]

\[ = \ln(5) \left[ \frac{4}{2} - \frac{1}{2} \right] = 2 \ln(5) \]

(ii) \[ f(x, y) = \int_1^4 \int_{-\frac{1}{4}}^{\frac{1}{4}} x\sqrt{1+xy} \, dy \, dx \]

\[ = \int_1^4 \left[ \int_{-\frac{1}{4}}^{\frac{1}{4}} x\sqrt{1+xy} \, dy \right] \, dx \]

\[ = \int_1^4 \left[ \left. \frac{2}{3} \left( 1+xy \right)^{3/2} \right|_{y=-\frac{1}{4}}^{y=\frac{1}{4}} \right] \, dx \]

\[ = \int_1^4 \left[ \frac{2}{3} (1+x(\frac{1}{4}))^{3/2} - \frac{2}{3} (1+x(-\frac{1}{4}))^{3/2} \right] \, dx \]

\[ = \frac{2}{3} \int_1^4 \, dx \]

\[ = \frac{2}{3} \left[ x \right]_1^4 \]

\[ = \frac{2}{3} [4-1] = \frac{2}{3} (3) = 2 \]
THIS PAGE IS FOR ROUGH WORK