Math 123, Week 3: 
Determinants

Section 1: Determinants

We have seen that the condition for invertibility of a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

was $ad - bc \neq 0$. By analogy with the real numbers, the condition $ad - bc \neq 0$ is enough to make sure that we do not “divide by zero” in our matrix algebra system.

This is quite a bit less transparent than we might have hoped. With real numbers, it was very easy to see if we were dividing by zero—there is a zero, we see it, we identify it, and we stay away from it. For matrices, however, this might not be obvious since none of the entries in the matrix need be zero, even for small matrices.

We now formally give this important form a proper name.

**Definition 1**

The **determinant** of a $2 \times 2$ matrix is given by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We may now state the following result.

**Theorem 1**

A $2 \times 2$ matrix $A$ is invertible if and only if

$$\det(A) = ad - bc \neq 0.$$
Example 1

Use the determinant to determine whether the following matrices are invertible:

\[ A = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}. \]

Solution: We have

\[
A = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-4)(2) = 11
\]

and

\[
B = \begin{vmatrix} -1 & 2 \\ 3 & -6 \end{vmatrix} = (-1)(-6) - (3)(2) = 0.
\]

Since \( \det(A) \neq 0 \), \( A \) is invertible, and since \( \det(B) = 0 \), \( B \) is not. Notice that we did not need to attempt to compute the inverses in order to determine whether the matrices were invertible. This will be a very useful feature when we generalized determinants to larger matrices.

Section 2: Higher Dimensions

Now consider determining when the general \( 3 \times 3 \) matrix

\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

is invertible. In order to compute this directly, we would need to perform Gaussian elimination on:

\[
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

At least in principle, this is something that we can do. The algebra, however, would become very tedious very quickly. If we are to believe the motivating example of \( 2 \times 2 \) matrices, this algebra would also be unnecessary. After all, we only need to determine whether we are dividing in the matrix algebra.
system, and determining that for $2 \times 2$ matrices required a simpler computation the full determination of the inverse. In fact, we only need to compute a single combination of the entries of the matrix.

It turns out that, for a general $3 \times 3$ matrix (and higher dimension), there is once again a single combination of the matrix entries which determines whether we are dividing by zero. This term, however, is quite large if written out explicitly and is cumbersome to work with. Rather, we will compute the term individually for given matrices using an inductive formula. We will see shortly that it is useful for many other computations as well.

We introduce the following.

**Definition 2**

Consider an $n \times n$ matrix $A$. Then:

1. The $(i, j)$-**minor** $M_{ij}$ of the matrix general $n \times n$ matrix $A$ is the determinant of the submatrix of $A$ produced by removing the $i^{th}$ row and $j^{th}$ column from $A$.

2. The $(i, j)$-**cofactor** is defined by $A_{ij} = (-1)^{i+j}M_{ij}$.

3. The **determinant** of $A$ is given by

   $\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} a_{ji} A_{ji}$

   where $i = 1, \ldots, n$ is fixed (and arbitrary).

The first thing we should notice is that these operations are defined inductively. The definitions of minors and cofactors require $(n-1) \times (n-1)$ submatrices of $A$. These terms are then used to define the determinant of an $n \times n$ matrix. It follows that, in order to compute the determinant of a $3 \times 3$ matrix, we will need to compute determinants of $2 \times 2$ matrices. The determinant of a $4 \times 4$ matrix depends on $3 \times 3$ matrix (which depends on $2 \times 2$ matrices), and so on.

Notice that the sign pattern of the cofactors obeys that patter of being positive if $i + j$ is even and odd if $i + j$ is odd. Consider the following example.
Example 2

Determine $M_{12}$, $A_{12}$, $M_{31}$, and $A_{31}$ for

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 5 & -3 \end{bmatrix}.$$ 

Solution: To determine $M_{12}$ we need to remove the first row and second column of $A$. We have that

$$M_{12} = \left| \begin{array}{cc} -1 & 1 \\ 1 & -3 \end{array} \right| = (-1)(-3) - (1)(1) = 2.$$ 

Since we have that $i + j = 3$ is odd, we have that $A_{12} = (-1)M_{12} = -2$.

To determine $M_{31}$ we need to remove the third row and the first column of $A$. We have that

$$M_{31} = \left| \begin{array}{cc} 1 & -2 \\ 0 & 1 \end{array} \right| = (1)(1) - (-2)(0) = 1.$$ 

Since we have that $i + j = 4$ is even, we have that $A_{31} = (1)M_{12} = 1$.

Now consider the determinant formula. This formula is a little daunting at first glance but there is a very distinctive pattern which will become second-nature after practice (although, unfortunately, there is still plenty of algebra to do):

1. Pick a row or a column in the matrix $A$. This is the $i = 1, \ldots, n$, in the formula and may be chosen arbitrarily.

2. For each component in the row/column, compute the product of the entry and the determinant of the submatrix with that row and column removed.

3. Determine the sign (i.e. $+1$ or $-1$) in the cofactor formula along the row/column.

4. Add or substract the results across each entry in the fixed row or column according to the sign pattern of the cofactors.
Note: The signs for the cofactors follow the following alternating pattern:

\[
\begin{pmatrix}
+ & - & + & \cdots \\
- & + & - & \cdots \\
+ & - & + & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Example 3

Determine the determinant of the matrix \( A \) given by

\[
A = \begin{bmatrix}
3 & 1 & -2 \\
-1 & 0 & 1 \\
1 & 5 & -3
\end{bmatrix}
\]

by expanding along the first row. Confirm the result by expanding along the second column.

Solution: The first row corresponds to the row with entries 3, 1, and \(-2\). By the formula, we have

\[
\begin{align*}
\det(A) &= (3) \begin{vmatrix} 0 & 1 & -3 \\ 1 & 1 & -3 \\ -1 & 1 & 5 \end{vmatrix} - (1) \begin{vmatrix} -1 & 1 & -3 \\ 1 & -1 & 5 \\ 3 & -2 & 1 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 0 & 1 \\ 1 & -3 & -3 \\ 3 & -2 & 1 \end{vmatrix} \\
&= (3)[(0)(-3) - (5)(1)] - (1)[(-1)(-3) - (1)(1)] + (-2)[(-1)(5) - (0)(1)] \\
&= 3(-5) - (1)(2) + (-2)(-5) \\
&= -7
\end{align*}
\]

The second column corresponds to the column with entries 1, 0, and 5. We have to be careful to apply the correct sign pattern for the cofactors. We have

\[
\begin{align*}
\det(A) &= -(1) \begin{vmatrix} -1 & 1 & -3 \\ 1 & -3 & 1 \\ 3 & -2 & 5 \end{vmatrix} + (0) \begin{vmatrix} 3 & -2 & 1 \\ 1 & -3 & -3 \\ 3 & -2 & 1 \end{vmatrix} - (5) \begin{vmatrix} 3 & -2 & 1 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} \\
&= -(1)[(-1)(-3) - (1)(1)] - (5)[(3)(1) - (2)(-1)] \\
&= -(1)(2) - (5)(1) \\
&= -7.
\end{align*}
\]
That was a little bit of work, for sure, but it was significantly easier than trying to invert the general form of the inverse $A^{-1}$ and seeing which term corresponded to dividing by zero. Remarkably, though, this is exactly the term which would fall out from such a computation in any dimension. We have the following very important result (which will unfortunately be beyond the scope of this course to prove, but I encourage you to check it with a computer for a few higher dimensions!).

**Theorem 2**

An $n \times n$ matrix is invertible if and only if $\det(A) \neq 0$.

**Example 4**

Determine whether the following matrix is invertible:

$$A = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 3 & -1 & 2 & 0 \\ 1 & 2 & -7 & 3 \\ -1 & 0 & 5 & 0 \end{bmatrix}.$$

**Solution:** This task seems overwhelming at first glance. In order to compute a $4 \times 4$ determinant, we need to compute $3 \times 3$ determinants, which required determination of $2 \times 2$ determinants. This is a lot of work in general, and work usually best left to a computer.

In this case, however, we have a shortcut. Recall that we can expand the determinant along any row or column we choose. In particular, we can choose the fourth column, which contains only one non-zero entry. This will simplify our analysis. We have

$$\det(A) = (-3) \begin{vmatrix} 1 & 0 & -5 \\ 3 & -1 & 2 \\ -1 & 0 & 5 \end{vmatrix}.$$  

where the sign has been chosen by the cofactor expansion. In general, this would still be a bit of work, but we can identify the two zeroes in
the second column to get

\[ \det(A) = (-3)(-1) \begin{vmatrix} 1 & -5 \\ -1 & 5 \end{vmatrix} \]
\[ = (-3)(-1)[(1)(5) - (-5)(-1)] \]
\[ = 0. \]

We can therefore determine that the matrix is not invertible.

Section 3: Elementary Matrixes

Consider the following definition.

**Definition 3**

An \( n \times n \) matrix \( E \) is an **elementary matrix** if it differs from the identity by a single row operation. Specifically, it differs from the identity matrix by one application of either: (1) switching rows; (2) scaling rows; and (3) adding rows.

Elementary matrices have the property that, for any matrix \( A \), the operation \( E \cdot A \) corresponds to the performing the respective elementary row operation on the matrix \( A \). Consider the following examples.

**Example 5**

Consider the following \( 3 \times 3 \) matrix:

\[ A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix}. \]

Determine the elementary matrices \( E_1, E_2, \) and \( E_3 \) corresponding respectively to (1) switching the second and third rows; (2) scaling the second row by 3; and (3) replacing the third row with the sum of the first and third row. Show that the matrices \( E_1 \cdot A, E_2 \cdot A, \) and \( E_3 \cdot A \) accomplish the corresponding row operation on the matrix \( A \).
Solution: The desired matrices are

\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\]

We can clearly see that

\[
E_1 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ -5 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}
\]

\[
E_2 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -15 & 3 & 6 \end{bmatrix}
\]

\[
E_3 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -4 & 1 & -1 \end{bmatrix}
\]

We will see that elementary matrices will allow many properties of determinants to be determined. First, however, we answer the more direct question of what their determinants are.

**Theorem 3**

Consider an elementary matrix \( E \). Then:

1. If \( E \) corresponds to the row operation of switching rows, then \( \det(E) = -1 \).
2. If \( E \) corresponds to the row operation of multiplying a single row by \( k \), then \( \det(E) = k \).
3. If \( E \) corresponds to the row operation of replacing any row with the sum of two other rows, then \( \det(E) = 1 \).

**Proof**

(1) By properties of determinants, we may expand upon any row or
column. If we expand upon the \( n - 2 \) rows which are not switched, and therefore correspond to the identity matrix, we have

\[
\det(E) = (1)^{n-2} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.
\]

(2) Expanding in any order or rows and columns we wish, we have

\[
\det(E) = (1)^{n-1}(k) = k.
\]

(3) If we expand upon the \( n - 2 \) rows which are not involved in the summation, we have either

\[
\det(E) = (1)^{n-2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}
\]

or

\[
\det(E) = (1)^{n-2} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}.
\]

In either case, we have \( \det(E) = 1 \).

Section 4: Properties of Determinants

It is important to consider how the value of the determinant changes as we perform different matrix operations. The foremost of these properties is matrix multiplication, which factors when we wish to determine the inverse of a matrix and also, as a result of the previous section, when we perform elementary row operations.

We have the following result.

**Theorem 4**

Consider two \( n \times n \) matrices \( A \) and \( B \). Then

\[
\det(A \cdot B) = \det(A) \cdot \det(B).
\]

We will unfortunately not have enough time to prove this result. For those motivated, consider using breaking the problem into cases, and for the cases
where $A$ is invertible, replace the matrix by the sequence of elementary matrices corresponding to the row reduction procedure. The result can be obtained by inductively multiplying the elementary matrices on the left. That is, start by showing $\det(E_1 \cdot B) = \det(E_1) \cdot \det(B)$, for the first row operation, and work left. It is a little bit of work, but also a worthwhile exercise.

Theorem 4 allows us to unlock many deeper properties of determinants. We have the following.

**Properties of Determinants**

Consider two $n \times n$ matrices $A$ and $B$. Then:

1. **Elementary row operations:** If $B$ and $A$ differ only by:
   - (a) two interchanged rows or interchanged columns, then $\det(B) = -\det(A)$.
   - (b) a single row (or column) of $B$ which is a multiple of $k$ of the corresponding row (or column) in $A$, then $\det(B) = k \cdot \det(A)$.
   - (c) a single row of $B$ which is the summation of two rows of $A$, then $\det(B) = \det(A)$.

2. **Transpose:** $\det(A) = \det(A^T)$

3. **Inverse:** if $A$ is invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$.

4. **Zero Row:** if $A$ contains a row (or columns) of zeros, then $\det(A) = 0$.

5. **Scalar Multiples:** if $A$ contains two rows (or columns) which are scalar multiples of one another, then $\det(A) = 0$.

**Proof**

**Elementary row operations:** Since elementary row operations can be performed by left multiplication of elementary matrices, we have $B = E \cdot A$ for some elementary matrix $E$. From Theorem 4, however, we have that $\det(B) = \det(E \cdot A) = \det(E) \cdot \det(A)$. The result follows from properties of $\det(E)$ (Theorem 3).
**Transpose:** The follows the cofactor expansion of the determinant. For instance, we obtain the identical expression when we expand the determinant along the $i$th row of $A$ as we do when we expand upon the $i$th column of $A$.

**Inverse:** By the inverse, we have that $A \cdot A^{-1} = I$. By Theorem 4, we have that $\det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$ while $\det(I) = 1$. It follows that $\det(A) = \frac{1}{\det(A^{-1})}$.

**Zero Row:** The is trivial by the cofactor expanding, where we expand along the row or column with only zeroes.

**Scalar Multiples:** This one is more complicated, but still follows from the cofactor expansion. In this case, however, we expand in such a way that the two rows which are scalar multiples are the last to be evaluated. Suppose that $j$th row is a multiple of $k$ of the $i$th row. Our final cofactors are then all of the form

$$\begin{vmatrix} a_{ip} & a_{iq} \\ a_{jp} & a_{jq} \end{vmatrix} = \begin{vmatrix} a_{ip} & a_{iq} \\ k \cdot a_{ip} & k \cdot a_{iq} \end{vmatrix} = k \cdot a_{ip} \cdot a_{iq} - k \cdot a_{ip} \cdot a_{iq} = 0$$

where $p$ and $q$ are arbitrary columns. Since the choice of columns was arbitrary, and every term in the cofactor expansion contains such a term, we have that $\det(A) = 0$.

**Example 6**

Determine the determinant of the following matrices:

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -2 \\ 1 & 5 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 1 & 12 \\ 0 & 0 & 0 \\ 8 & -3 & 2 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 2 & 6 & -4 \\ 5 & -1 & -6 \\ -3 & -9 & 6 \end{bmatrix}$$

**Solution:** We earlier used the cofactor expansion to determine that
\[ \det(A) = -7. \] We could evaluate the determinant of \( B \) directly; however, it is quicker to notice that the matrix was obtained by interchanging the first and second rows, and then multiplying the first row by \(-2\). Explicitly, we have

\[
\begin{vmatrix}
1 & 0 & -1 \\
3 & 1 & -2 \\
1 & 5 & -3
\end{vmatrix} = (-1) \cdot 
\begin{vmatrix}
-1 & 0 & 1 \\
3 & 1 & -2 \\
1 & 5 & -3
\end{vmatrix}
\]

\[ = (-1) \cdot (-1) \cdot 
\begin{vmatrix}
3 & 1 & -2 \\
-1 & 0 & 1 \\
1 & 5 & -3
\end{vmatrix}
\]

\[ = (-1) \cdot (-1) \cdot (-7) = -7. \]

For \( C \), again, we could compute it directly, but we could also notice that the matrix can be obtained by replacing the second row in \( B \) with the second row minus three times the first row, and replacing the third row of \( B \) with the third row minus the first row. Notice that this sequence of row operations can be carried out as multiplying the first row by \(-3\) (or \(-1\)), then adding the rows, then multiplying the first row again by \(-3\) (or \(-1\)) so that the determinant does not change. We have

\[
\begin{vmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 5 & -2
\end{vmatrix} = \begin{vmatrix}
1 & 0 & -1 \\
3 & 1 & -2 \\
1 & 5 & -3
\end{vmatrix} = \begin{vmatrix}
1 & 0 & -1 \\
3 & 1 & -2 \\
1 & 5 & -3
\end{vmatrix} = -7.
\]

We can quickly determine that \( \det(D) = 0 \) because there is a row of zeroes. We can also see that \( \det(F) = 0 \) by noticing that the third row is \(-3/2\) times the first row.

**Section 5: Cramer’s Rule**

We have already determined two methods by which to solve a linear system of equations \( A \cdot \mathbf{x} = \mathbf{b} \). We now introduced a third which is occasionally advantageous.

**Cramer’s Rule**
Consider a linear system \( A \cdot x = b \) where \( A \) is an \( n \times n \) invertible matrix. Then, for every \( i = 1, \ldots, n \), we have
\[
x_i = \frac{\det(A_i)}{\det(A)}
\]
where \( A_i \) is the matrix \( A \) with the \( i \)th column replaced by the vector \( b \).

**Note:** In general, Cramer’s Rule is no more efficient than any other method for solving linear systems; however, it does have the distinct advantage of being able to solve for a single variable \( x_i \) at a time. It also allows easy computation of fractional answers. Fractions were often very difficult to handle when solving by Gaussian row reduction.

**Proof**

We will use basic properties of determinants. We start by introducing a matrix
\[
D_i = \begin{bmatrix}
a_{11} & \cdots & a_{1i}x_i & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2i}x_i & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & \cdots & a_{ni}x_i & \cdots & a_{nn}
\end{bmatrix}
\]
That is, \( D_i \) is the matrix \( A \) with the variable we are attempting to solve for \((x_i)\) multiplied throughout the \( i \)th column. By determinant properties, we have
\[
\det(D_i) = \det(A) \cdot x_i.
\]

However, we also have the property that the determinant is not altered by adding multiples of other rows. We therefore introduce the matrix
\[
A_i = \begin{bmatrix}
a_{11} & \cdots & b_1 & \cdots & a_{1n} \\
a_{21} & \cdots & b_2 & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & \cdots & b_n & \cdots & a_{nn}
\end{bmatrix} = \begin{bmatrix}
a_{11} & \cdots & \sum_{j=1}^n a_{1j}x_j & \cdots & a_{1n} \\
a_{21} & \cdots & \sum_{j=1}^n a_{2j}x_j & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & \cdots & \sum_{j=1}^n a_{nj}x_j & \cdots & a_{nn}
\end{bmatrix}.
\]
It follows that \( \det(A_i) = \det(D_i) \) so that

\[
\det(A_i) = \det(A) \cdot x_i \implies x_i = \frac{\det(A_i)}{\det(A)}.
\]

**Example 7**

Use Cramer’s Rule to determine the value of \( z \) for the system:

\[
\begin{align*}
    x - 3z & = 1 \\
    2x + y - z & = -2 \\
    -5x + y + 2z & = 2.
\end{align*}
\]

**Solution:** The system can be written in matrix form as

\[
\begin{bmatrix}
    1 & 0 & -3 \\
    2 & 1 & -1 \\
    -5 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
= \begin{bmatrix}
    1 \\
    -2 \\
    2
\end{bmatrix}.
\]

The matrices we need are

\[
A = \begin{bmatrix}
    1 & 0 & -3 \\
    2 & 1 & -1 \\
    -5 & 1 & 2
\end{bmatrix} \quad \text{and} \quad A_z = \begin{bmatrix}
    1 & 0 & 1 \\
    2 & 1 & -2 \\
    -5 & 1 & 2
\end{bmatrix}
\]

where the matrix \( A_z \) as the third column, corresponding to the coefficients of \( z \), replaced by the vector \( b = (1, -2, 2) \). We have

\[
\det(A) = (1) \left| \begin{array}{cc}
    1 & -1 \\
    2 & 1
\end{array} \right| + (-3) \left| \begin{array}{cc}
    2 & 1 \\
    -5 & 1
\end{array} \right| = (1)(3) + (-3)(7) = -18
\]

and

\[
\det(A_z) = (1) \left| \begin{array}{cc}
    1 & -2 \\
    2 & 1
\end{array} \right| + (1) \left| \begin{array}{cc}
    2 & 1 \\
    -5 & 1
\end{array} \right| = (1)(4) + (1)(7) = 11.
\]
It follows by Cramer’s Rule that the value of $z$ is
\[ z = \frac{\det(A_z)}{\det(A)} = -\frac{11}{18}. \]

Section 6: The Adjugate (Adjoint)

We now introduce a matrix which has applications for determining inverses of a matrix.

**Definition 4**

The **adjugate** (or **classical adjoint**) of an invertible $n \times n$ matrix $A$ is the matrix with entries
\[ [\text{adj}(A)]_{ij} = A_{ji} \]
where $A_{ij}$ is the $(i, j)$-cofactor of $A$.

The adjugate is transpose of the cofactor matrix since its entries are the cofactors of the matrix $A$. It is important to remember that the cofactors obey the alternating sign pattern. The adjugate was previously more commonly known as the adjoint; however, this term has in recent years become more commonly associated with another matrix important to the study of linear algebra. Many old textbooks, however, still refer to this quantity as the adjoint, and the shortform $\text{adj}(A)$ is no coincidence.

**Example 8**

Determine the adjugate of the matrix
\[ A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix}. \]
Solution: We have
\[
\text{adj}(A) = \begin{bmatrix}
1 & -1 & -2 & -1 & 2 & 1 \\
1 & 2 & -5 & 2 & -5 & 1 \\
0 & -3 & 1 & -3 & 1 & 0 \\
1 & -1 & -5 & 2 & 1 & 1 \\
\end{bmatrix}^T
\]
\[
T = \begin{bmatrix}
3 & 1 & 7 \\
-3 & -13 & -1 \\
3 & -5 & 1 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
3 & -3 & 3 \\
1 & -13 & -5 \\
7 & -1 & 1 \\
\end{bmatrix}
\]

The adjugate is pretty unexceptional, aside from the following property.

Theorem 5
Consider an \( n \times n \) invertible matrix \( A \). Then the inverse of \( A \) is given by
\[
A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A).
\]

Proof
This is actually a direct application on Cramer’s Rule. To determine the inverse of a matrix \( A \), we need to find a \( B \) which satisfies
\[
A \cdot B = I.
\]
We can set this up as a system of \( n \) linear systems of the form
\[
A \cdot B_j = I_j
\]
where \( B_j \) and \( I_j \) are the \( j \)th columns of \( B \) and \( I \), respectively. By
Cramer’s Rule we have that the entries of $B$ satisfy

$$b_{ij} = \frac{\det(A_i^{(j)})}{\det(A)}$$

where $A_i^{(j)}$ is the matrix $A$ with the $i$th column replaced by the $j$th column of $I$ (which, of course, has a one in the $j$th entry and is zero otherwise). Notice that this matrix has the one from the identity matrix in the $j$th row and $i$th column. By expanding upon this column, we can quickly determine that

$$\det(A_i^{(j)}) = A_{ji}$$

where $A_{ji}$ is $(j,i)$-cofactor of $A$. The formula follows by the definition of the transpose.

**Example 9**

Use the adjugate to determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix}.$$  

**Solution:** We already have the adjoint from the previous example, and the determinant from the example prior to that. We just need to apply the formula. We have

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

$$= -\frac{1}{18} \begin{bmatrix} 3 & -3 & 3 \\ 1 & -13 & -5 \\ 7 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{18} \begin{bmatrix} -3 & 3 & -3 \\ -1 & 13 & 5 \\ -7 & 1 & -1 \end{bmatrix}.$$
1. Compute the determinant of the following matrices:

\[
A = \begin{bmatrix} 2 & 0 & 1 \\ 7 & 1 & 3 \\ -4 & -1 & 1 \end{bmatrix}, \quad
B = \begin{bmatrix} 2 & 0 & 1 \\ 5 & 1 & 2 \\ -4 & -1 & 1 \end{bmatrix}, \quad
C = \begin{bmatrix} 3 & 0 & -3 & 0 \\ 7 & -3 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 3 & -2 & 4 & 1 \end{bmatrix}
\]

2. Consider a 4 × 4 matrix \( A \). State the elementary matrices corresponding to the following operations:

   (a) Switching the second and fourth rows.
   (b) Multiplying the third row by \(-3\).
   (c) Replacing the first row with the sum of the first and fourth rows.

3. Since elementary row operations change the determinant of a matrix in regular ways, and the determinant of the identity matrix is one, we can compute the determinant of a matrix by perform the row operations required to transform the matrix to the identity matrix (i.e. performing Gauss-Jordan row reduction). Use this method to compute the determinants of the following matrices:

\[
A = \begin{bmatrix} -3 & 2 & -4 \\ 1 & 1 & -2 \\ -1 & 3 & -1 \end{bmatrix}, \quad
\text{and} \quad
B = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 3 & 0 \end{bmatrix}.
\]

4. Use Cramer’s Rule to determine the value of \( y \) in the following system:

\[
\begin{align*}
-2x + y + 5z &= 1 \\
x - 2y + z &= 0 \\
2x + 2y - z &= -3.
\end{align*}
\]

5. Compute the adjugate for the following matrix, and then use it to determine the inverse:

\[
A = \begin{bmatrix} -3 & -1 & -1 \\ 1 & 1 & -2 \\ 2 & 1 & 2 \end{bmatrix}, \quad
\text{and} \quad
B = \begin{bmatrix} 1 & -1 & -5 \\ 1 & 0 & -1 \\ 1 & 3 & 0 \end{bmatrix}.
\]