1. Definitions:

(a) A linear system $A \cdot x = b$ is **consistent** if...

(b) A set $W$ is a **subspace** of $\mathbb{R}^n$ if...

(c) The **null space** of an $m \times n$ matrix $A$ is...

2. True/False:

(a) The inverse of an invertible $n \times n$ matrix $A$ is given by $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$.
   [ True / False ]

(b) Suppose $B$ is a basis of an 11-dimensional subspace of $\mathbb{R}^{18}$. Then $B$ contains 18 vectors.
   [ True / False ]

(c) The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$ (rather than $\mathbb{R}^n$).
   [ True / False ]

(d) For an $m \times n$ matrix $A$, the set of vectors $v \in \mathbb{R}^n$ such that $A \cdot v = b$ for $b \neq 0$ is a subspace of $\mathbb{R}^n$.
   [ True / False ]
3. Short Answer: Consider the following $3 \times 3$ matrix:

$$A = \begin{bmatrix} 4 & -1 & -3 \\ 5 & -2 & -3 \\ 1 & -1 & 0 \end{bmatrix}.$$

(a) Show that $v_1 = (1, 1, 1)$ is in the nullspace of $A$, i.e. $v \in \text{null}(A)$.

(b) Show that $v_2 = (1, 2, 1)$ is an eigenvector of $A$. What is the corresponding eigenvalue?

(c) Show that $b = (1, -1, 0)$ is not in the column space of $A$, i.e. $b \not\in \text{col}(A)$.

(d) Based on (c), what can be concluded about solutions, if any, of the linear system

$$\begin{bmatrix} 4 & -1 & -3 \\ 5 & -2 & -3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}?$$

[Hint: You do not need to have solved (c) to answer this!]
4. Linear Systems:

(a) Consider the following linear system:

\[
\begin{bmatrix}
-1 & 0 & -1 \\
3 & -2 & 1 \\
2 & 1 & 3 + k
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-3 \\
-1
\end{bmatrix}
\]

Determine the values of \(k \in \mathbb{R}\) for which (i) there is no solution, (ii) there is a unique solution, and (iii) there are an infinite number of solutions.

(b) Given that

\[
A = \begin{bmatrix}
3 & -2 & 1 \\
0 & 2 & 1 \\
2 & -1 & 2
\end{bmatrix} \implies A^{-1} = \frac{1}{17} \begin{bmatrix}
3 & 5 & 4 \\
-2 & 8 & 3 \\
4 & 1 & -6
\end{bmatrix}
\]

find the solution \((x, y, z)\) of the following linear system:

\[
\begin{align*}
3x - 2y + z &= 1 \\
2y + z &= -1 \\
2x - y - 2z &= 2
\end{align*}
\]
5. Determinants and Inverses:
   Consider the matrix
   \[ A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & -1 & 0 \\ 0 & -1 & 3 \end{bmatrix}. \]

   (a) Find the determinant of \( A \), \( \det(A) \).
   
   [2]  

   (b) Find the inverse of \( A \), \( A^{-1} \). [Note: You may use any method you wish.]
   
6. Matrix Spaces:  
   Consider the following matrix and its row-reduced echelon form (right):
   \[
   \begin{bmatrix}
   1 & -1 & 1 & 1 & 2 \\
   -1 & 1 & -1 & -1 & -2 \\
   1 & -1 & 1 & 2 & 1 \\
   \end{bmatrix} \implies \begin{bmatrix}
   1 & -1 & 1 & 0 & 3 \\
   0 & 0 & 0 & 1 & -1 \\
   0 & 0 & 0 & 0 & 0 \\
   \end{bmatrix}.
   \]

   State a basis for the following:

   (a) The row space of \( A \).
   
   [1]  

   (b) The column space of \( A \).
   
   [1]  

   (c) The null space of \( A \).
   
   [1]
7. Eigenvalues and Eigenvectors:

Determine the eigenvalues and eigenvectors of the following matrix:

\[ A = \begin{bmatrix} -2 & -3 \\ 8 & 8 \end{bmatrix}. \]

[**Hint:** You should find two real, distinct roots of the characteristic equation.]

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8. Theory:

Consider three vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3 \). Prove that, if \( \mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \), then \( \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \). **[Hint:** Note that the assumption \( \mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \) implies that there are \( c_1, c_2 \in \mathbb{R} \) such that \( \mathbf{v}_3 = c_1 \cdot \mathbf{v}_1 + c_2 \cdot \mathbf{v}_2 \). Use this to prove any vector \( \mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) also satisfies \( \mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \).]**
THIS PAGE IS FOR ROUGH WORK