In order to take a step forward in our understanding of vector and matrix operations, we will have to first take a step backward—or rather, sideways. We will briefly remove ourselves from the consideration of any particular application, and introduce a general mathematical notion which will prove useful in many applications throughout the rest of the course.

Consider the following definition.

**Definition 1**

Consider a nonempty set $V$. We will say that $V$ is a vector space if there is an addition ($+$) and scalar multiplication ($\cdot$) operation which satisfies the following axioms:

1. **Addition:**
   
   (a) $v + w = w + v$ for all $v, w \in V$.
   
   (b) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
   
   (c) There is a $0 \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.
   
   (d) For each $v \in V$, there is a $-v \in V$ such that $v + (-v) = 0$.

2. **Scalar Multiplication:**
   
   (a) $c \cdot (d \cdot v) = (cd) \cdot v$ for all $c, d \in \mathbb{R}$ and $v \in V$.
   
   (b) $1 \cdot v = v$ for all $v \in V$.

3. **Distributivity:**
   
   (a) $c \cdot (v + w) = c \cdot v + c \cdot w$ for all $c \in \mathbb{R}$, and $v, w \in V$.
   
   (b) $(c + d) \cdot (v) = c \cdot v + d \cdot v$ for all $c, d \in \mathbb{R}$ and $v \in V$. 
Vector spaces are the foundational concept upon which all others in linear algebra are built. These axioms might seem redundant, since we have considered most of them when defining vector operations earlier in the course, but the applications of such spaces extend significantly farther than what we have seen.

Example 1

The most common vector spaces are the sets \( V = \mathbb{R}^2 \) and \( V = \mathbb{R}^3 \), which consist respectively of vectors of the form \( \mathbf{v} = (v_1, v_2) \) and \( \mathbf{v} = (v_1, v_2, v_3) \), where we take the standard vector addition and scalar multiplication operations we have already defined.

In fact, we can generalized to the vector space \( V = \mathbb{R}^n \) (i.e. \( R^n \)), which contains vectors \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \). Regardless of the value of \( n \) (even allowing it to be infinity), the vector space axioms are very easily shown to be satisfied. Note that the zero vector is \( \mathbf{0} = (0, \ldots, 0) \) and the additive inverse is \( -\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \).

We should pause to note that being a vector space depends on both the elements in the set and also on how the addition and multiplication operations defined. In particular, we need not define these operations in a sensible way, and if we are not careful, we may break the vector space axioms. Consider the following example.

Example 2

Show that the set \( V = \mathbb{R}^2 \) is not a vector space when taken with the following operations:

\[
\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 - w_2)
\]
\[
c \cdot \mathbf{v} = (cv_1, cv_2).
\]

Solution: We need to show that at least one of the axioms is violated. We start from the first and work our way down. Condition 1.(a) gives

\[
\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 - w_2)
\]
\[
\mathbf{w} + \mathbf{v} = (w_1 + v_1, w_2 - v_2).
\]
In fact, we have that $v + w \neq w + v$ for any vector not satisfying $v_2 = w_2$. For example, for the vectors $v = (1, 0)$ and $w = (0, 1)$, we have $v + w = (1, -1)$ and $w + v = (1, 1)$ so that $v + w \neq w + v$. Even without checking the rest of the axioms, we have that the set $V = \mathbb{R}^2$ taken with these vectors operations is not a vector space.

Note: Vector spaces need not have anything to do with conventional vectors at all. For instance, let $V$ be the set of continuous functions on the real number line. Aside from the conceptual abstract of calling functions “vectors”, it should not take much convincing that the axioms are satisfied for the standard definitions of addition and multiplication of real numbers. For example, we clearly have that for any functions $f, g \in V$ that

1. $f(x) + g(x) = g(x) + f(x)$

2. $f(x) + 0 = 0 + f(x)$ where $0 \in V$ is the zero function (which is trivially continuous and so belongs to $V$)

3. $f(x) + (-f(x)) = (-f(x)) + f(x) = 0$ (so $-f(x)$ is the additive inverse of $f(x)$)

4. etc.

We will not deal with vector spaces more general than $\mathbb{R}^n$ in this course, but it is important to realize that much of the machinery we will develop can be applied to more general constructs. In particular, we can talk about a basis of any vector space.

Section 2: Subspaces

We will spend the majority of our time considering the vector space $V = \mathbb{R}^n$ with the standard vector addition and scalar multiplication operations. This space is familiar in many senses, but also contains a significant amount of subtlety when we begin to dig around the corners. Consider the following definition.
Definition 2
Consider a vector space \( V \). We will say that \( W \) is a \textbf{subspace} of \( V \) if:

1. For any \( v \in W \) we have \( v \in V \) (i.e. \( W \subseteq V \));
2. For any \( c \in \mathbb{R} \) and any \( v \in W \), we have \( c \cdot v \in W \); and
3. For any \( v, w \in W \), we have \( v + w \in W \).

Our convention will be to set \( V = \mathbb{R}^n \) so that \( W \subseteq \mathbb{R}^n \). Property 1. will therefore be satisfied by any set of vectors with the correct number of coordinates. Properties 2. and 3., respectively, indicate that subspaces are \textbf{closed under multiplication} and \textbf{closed under addition}. That is, we cannot take an element in the set and scale it to produce something not in the set. Similarly, we cannot take two elements in the set, and add them to obtain an element not in the set.

Consider the following examples.

Example 3
Show that the set of vectors of the form \( S = \{(s, t, s+t) \in \mathbb{R}^3 \mid s, t \in \mathbb{R}\} \) is a subspace of \( \mathbb{R}^3 \).

\textbf{Solution:} It is clear that every vectors of the form \((s, t, s+t)\) is in \( \mathbb{R}^3 \) so property 1. is trivially satisfied. To check property 2., we consider an arbitrary constant \( c \in \mathbb{R} \) and vector \( v \in S \) and check
\[
c \cdot v = c \cdot (s, t, s+t) = (cs, ct, cs + ct).
\]
We can see that this vector is still in the right form, taking \( \tilde{s} = cs \) and \( \tilde{t} = ct \), since this gives \( c \cdot v = (\tilde{s}, \tilde{t}, \tilde{s}+\tilde{t}) \). It follows that \( c \cdot (s, t, s+t) \in S \).

We now consider two vectors \( v_1, v_2 \in S \). We have
\[
v_1 + v_2 = (s_1, t_1, s_1 + t_1) + (s_2, t_2, s_2 + t_2) \\
= (s_1 + s_2, t_1 + t_2, s_1 + s_2 + t_1 + t_2) \\
= (\tilde{s}, \tilde{t}, \tilde{s} + \tilde{t})
\]
where \( \tilde{s} = s_1 + s_2 \) and \( \tilde{t} = t_1 + t_2 \), respectively. Again, this is in the correct form so that \( v_1 + v_2 \in S \). It follows that \( S \) is a subspace of \( \mathbb{R}^3 \).
Example 4

Show that \( W = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\} \) is a subspace of \( V = \mathbb{R}^2 \) and that \( W^* = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 2\} \) is not.

Solution: Consider the set of points \((x, y) \in \mathbb{R}^2\) satisfying \( x + 2y = 0 \). We have that \( c \cdot (x, y) = (cx, cy) \) and need to determine if this points satisfies the given relationship. In fact, we have that

\[
(cx) + 2(cy) = c(x + 2y) = 0
\]

since \( x + 2y = 0 \). It follows that \( c \cdot (x, y) \in W \).

Now consider \((x_1, y_1) \in W\) and \((x_2, y_2) \in W\), so that \( x_1 + 2y_1 = 0 \) and \( x_2 + 2y_2 = 0 \). We want to determine if \( x + y = (x_1 + y_1, x_2 + y_2) \) satisfies the relationship. We have that

\[
(x_1 + x_2) + 2(y_1 + y_2) = (x_1 + 2y_1) + (x_2 + 2y_2) = 0.
\]

It follows that \((x_1, y_1) + (x_2, y_2) \in W\), and we are done.

Now consider the set of points \((x, y) \in \mathbb{R}^2\) satisfying \( x + 2y = 2 \). We want to check whether \( c \cdot (x, y) = (cx, cy) \) satisfies the relationship. We have that

\[
(cx) + 2(cy) = c(x + 2y) = 2c
\]

since \( x + 2y = 2 \). In fact, we do not have the relationship satisfied unless \( c = 1 \). This is enough to ruin the argument. For example, we could take \( c = 2 \) and \((x, y) = (2, 0)\) to get \( c \cdot (x, y) = 2 \cdot (2, 0) = (4, 0) \) which implies \( x + 2y = 4 + 2(0) = 4 \neq 2 \). It follows that \( c \cdot (x, y) \not\in W^* \) so that \( W^* \) is not a subspace of \( \mathbb{R}^2 \).

To consider what has happened in this example, consider the following geometric figure:
We can see that both $W$ and $W^*$ lie within $V = \mathbb{R}^2$ and that both sets correspond to straight lines. In order to satisfy $c \cdot v \in W$ for any $c \in \mathbb{R}$, however, we must be able to move around in our space $W$ as far along straight lines as possible without leaving the set. A consequence of this is, at the very least, that subspaces must go through $(0,0)$. There are many other intuitive (and not so intuitive) consequences which we consider shortly.

Section 3: Spanning Sets

To motivate our consideration of how vector spaces apply to the matrix systems, we will consider the mechanics of Gaussian elimination in more detail. Consider the following linear system:

$$\begin{cases} 3x - y + 4z = 0 \\ x - y = 0 \\ x - 2y - 2z = 0. \end{cases}$$

(1)

Note that this type of linear system is called homogeneous because there are no terms which are independent of the variables of interest ($x$, $y$, and $z$). Homogeneous systems are easy to identify because they can be put into the form $A \cdot x = 0$ where $0$ is the vector of all zeroes. After a few (omitted) steps we arrive at the following row-reduced echelon form:

$$\begin{bmatrix} 3 & -1 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & -2 & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Of course, we can now read off the solution by making the substitution
\( z = t \) and solving for \( x \) and \( y \); however, we want to make considerations
on the process of performing elementary row operations themselves. Notice
that each elementary row operation can be imagined as an operation on
vectors by rewriting the matrix \( A \) as being composed of three row vectors
\( \mathbf{a}_1, \mathbf{a}_2, \) and \( \mathbf{a}_3. \) In other words, we could write
\[
A = \begin{bmatrix}
3 & -1 & 4 \\
1 & -1 & 0 \\
1 & -2 & -2
\end{bmatrix} = \begin{bmatrix}
\mathbf{a}_1 \\
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix}
\]
where \( \mathbf{a}_1 = (3, -1, 4), \mathbf{a}_2 = (1, -1, 0), \) and \( \mathbf{a}_3 = (1, -2, -2). \) The only way
we could construct a new vector \( \mathbf{a} \) through elementary row operations is by
scaling rows or adding rows together. It should not take much justification
to convince ourselves that every new row we could possibly encounter can
therefore be written in the form
\[
\mathbf{a} = c_1 \cdot \mathbf{a}_1 + c_2 \cdot \mathbf{a}_2 + c_3 \cdot \mathbf{a}_3
\]
for some real values \( c_1, c_2, c_3 \in \mathbb{R}. \) A summation of this form is special
enough in the theory of linear algebra that it gets a special name.

**Definition 3**

Given a set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \), a summation of the form
\[
\sum_{i=1}^{m} c_i \cdot \mathbf{v}_i
\]
for real constants \( c_i \in \mathbb{R} \) is called a *linear combination* of the set.

It is clear that (2) is a linear combination of the vectors \( \mathbf{a}_1, \mathbf{a}_2, \) and \( \mathbf{a}_3. \)
What is less clear is what this set looks like. For example, we might wonder
whether a given vector can be attained through a linear combination of the
set \( \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \), or what the set of all linear combination of the vectors looks
like geometrically.

To ground discussion on these topics, we introduce the following.

**Definition 4**

Consider a set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \) where \( \mathbf{v}_i \in \mathbb{R}^n. \) Then the
span of $S$ is defined as
\[
\text{span} (S) = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_m \mathbf{v}_m \text{ for some } c_1, \ldots, c_m \in \mathbb{R} \}.
\]
In other words, the span is the set of all vectors which can be reached by a linear combination of the vectors in the set.

It turns out that we already have all of the tools required to determine whether a given vector is in the span of the set of another vector. Consider the following examples.

**Example 5**

Consider the linear system (1). Determine whether it is possible to attain the following rows as a result of performing elementary row operations on the rows of the system:

\[
\begin{align*}
(\ast) \quad x + y + z &= 0 \\
(\ast\ast) \quad y + 2z &= 0.
\end{align*}
\]

**Solution:** Since new rows are produced as linear combinations of old rows, we need to check whether

\[c_1 \cdot \mathbf{a}_1 + c_2 \cdot \mathbf{a}_2 + c_3 \cdot \mathbf{a}_3 = \mathbf{a}\]

where $\mathbf{a}_i$ are the coefficient vectors of the original system, and $\mathbf{a}$ is the coefficient vector of the row we are trying to obtain. If we can find values $c_i$ satisfying this equation, the row can be obtained through elementary row operations; otherwise, it cannot.

Taking $\mathbf{a} = (1, 1, 1)$ corresponding to $(\ast)$, we have

\[
c_1 \cdot \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

We quickly recognize that this is equivalent to the following linear sys-
Performing Gaussian row reduction (steps omitted), we have
\[ \begin{bmatrix} 3 & 1 & 1 \\ -1 & -1 & -2 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

The last line indicates the system is inconsistent so that not values \( c_1 \), \( c_2 \), and \( c_3 \) can be found. It follows that the row (*) may not be obtained through any sequence of elementary row operations.

Now consider \( \mathbf{a} = (0, 1, 2) \), corresponding to (**). We have
\[ \begin{bmatrix} 3 & 1 & 1 \\ -1 & -1 & -2 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \]

Row reduction yields
\[ \begin{bmatrix} 3 & 1 & 1 \\ -1 & -1 & -2 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \]

This may not be as pretty as we would like, but it is a consistent system. In fact, there are an infinite number of possibilities. If we set \( c_3 = t \) (i.e. since the third column corresponds to a free variable), we have that \( c_1 = \frac{1}{2} + \frac{1}{2} \cdot t \) and \( c_2 = -\frac{3}{2} - \frac{5}{2} \cdot t \). A valid solution is attained for every value of \( t \). For example, if we choose \( t = 1 \), we obtain \( c_1 = 1 \), \( c_2 = -4 \), and \( c_3 = 1 \). It can easily be verified that
\[ (1) \cdot \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + (-4) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (1) \cdot \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \]

This corresponds to the fact that, in the original system (1), if we take one times the first and third row, and minus four times the second row, we obtained the desired row (**), but no such combination will ever produce the row (*).
We briefly pause to discuss the geometry of linear combinations and spanning sets. We saw in the previous example that, given a set of vectors, it was possible that certain vectors could be obtained as linear combinations of the set while others could not. We might wonder what exactly the set of all vectors attainable through linear combinations looks like.

Since the vectors in this example were all three-dimensional, it is possible to visualize this set. It corresponds to the following picture:

![Visualization of linear combinations and spanning sets](image)

The vectors $a_1 = (3, -1, 4)$, $a_2 = (1, -1, 0)$, and $a_3 = (1, -2, -2)$ are the overlain in red, blue, and green, respectively.

We make a couple of notes based on this picture. The first is that the span of $a_1$, $a_2$, and $a_3$ is not all of $\mathbb{R}^3$. This is why we were not able to attain the vector $(1, 1, 1)$ through a linear combination of these vectors (although $(0, 1, 2)$ was on this surface). Secondly, we notice that the shape spanned by these vectors is a plane. It is as if we have carved out a two-dimensional subdomain in our three-dimensional space. We will formalized this concept in a few lectures.

We pause to connect the notion of spanning sets to vector spaces.

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**Theorem 1**

For any set of vectors $S = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$, we have that $W = \text{span}(S)$ is a subspace of $\mathbb{R}^n$. Furthermore, every subspace $W$ of $\mathbb{R}^n$ can be expressed as $W = \text{span}(S)$ for some finite set $S = \{v_1, \ldots, v_m\}$ where $m \leq n$. 

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Proof

Take any two vectors \( v \) and \( w \) in the span(\( S \)) where \( S = \{ v_1, \ldots, v_m \} \).
It follows that we have
\[
\begin{align*}
v &= c_1 \cdot v_1 + \cdots + c_m \cdot v_m \\
w &= c'_1 \cdot v_1 + \cdots + c'_m \cdot v_m.
\end{align*}
\]

It follows that we have
\[
v + w = (c_1 + c'_1) \cdot v_1 + \cdots + (c_m + c'_m) \cdot v_m
\]
so that \( v + w \in \text{span}(S) \). We also clearly have that, for any constant \( c \in \mathbb{R} \),
\[
c \cdot v = (c \cdot c_1) \cdot v_1 + \cdots + (c \cdot c_m) \cdot v_m
\]
so that \( c \cdot v \in \text{span}(S) \). It follows that the \( \text{span}(S) \) is a vector space.

Now suppose \( W \subseteq \mathbb{R}^n \) is a subspace and choose \( v_1 \in W \) arbitrarily.
We will build the set \( \{ v_1, \ldots, v_m \} \). Consider a vector \( v_2 \in W \). If
\( v_2 = c \cdot v_1 \), we do not include the vector, since it can obtain it by the scalar multiplication property of the subspace \( W \); otherwise we keep it.
Now consider a vector \( v_3 \in W \). In this case, we keep the vector so long as
\[
c_1 \cdot v_1 + c_2 \cdot v_2 \neq v_3
\]
for any \( c_1, c_2 \in \mathbb{R} \). And so on. We keep each successive vector and add it to the set so long as
\[
c_1 \cdot v_1 + \cdots + c_{m-1} \cdot v_{m-1} \neq v_m
\]
for any \( c_1, \ldots, c_{m-1} \in \mathbb{R} \).

Suppose that this process does not terminate. Then, at some point, we would have
\[
c_1 \cdot v_1 + \cdots + c_n \cdot v_n \neq v_{n+1}
\]
for any \(c_1, \ldots, c_n \in \mathbb{R}\), where \(n\) is the number of coordinates of vector \(v_i\). This can be written as the linear system

\[
\begin{bmatrix}
v_1 & \cdots & v_n \\
\vdots & & \vdots \\
c_n
\end{bmatrix}
\begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix} = v_{n+1}
\]

where the vectors \(v_i\) are column vectors. The surprising observation is that this system has a solution—specifically, we must be able to reduce the left-hand side to the identity matrix through Gaussian elimination. If we were not able to accomplish this, then the homogeneous system

\[
\begin{bmatrix}
v_1 & \cdots & v_n \\
\vdots & & \vdots \\
c_n
\end{bmatrix}
\begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix} = 0
\]

would reduce through Gaussian elimination to a system with a free parameter. Suppose, without loss of generality, that this free parameter corresponds to \(c_n\) so that we have

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & a_1 & 0 \\
0 & 1 & \cdots & 0 & a_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a_{n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

Then can then write \(c_n = t\) and \(c_i = -t \cdot a_i\) for \(i = 1, \ldots, n-1\). For \(t = 1\), we have

\[-a_1 \cdot v_1 - \cdots - a_{n-1} \cdot v_{n-1} + v_n = 0\]

so that

\[v_n = a_1 \cdot v_1 + \cdots + a_{n-1} \cdot v_{n-1}\]

This violates the assumption that \(v_n\) could not be written as a linear combination of the vectors \(\{v_1, \ldots, v_{n-1}\}\). It follows that the process of adding vectors must terminate at some point, and we need at most \(n\) vectors in the spanning set \(\{v, \ldots, v_n\}\).

This result was a handful, but it will outside some of the important techniques for establishing the how spanning sets work. Importantly, we
now know that the span of a subset of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, and that every subspace of $\mathbb{R}^n$ can be written as the span of some finite subset of vectors in $\mathbb{R}^n$. We can think of this set of vectors as generators of the vector space. We will discuss properties such as the uniqueness of such a generating set, and its minimal size, in a future lecture.

For now, consider the following example.

**Example 6**

Show that the set $S = \{(1, 0, 1), (2, -1, 0), (-1, 1, 2)\}$ spans $\mathbb{R}^3$.

**Solution:** We need to establish that, for an arbitrary vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, we may reach the vector through a linear combination of the vectors in $S$. This corresponds to

$$
\begin{bmatrix}
1 & 2 & -1 \\
0 & -1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= 
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}.
$$

This is a linear system which we can solve by Gaussian elimination. Omitting steps, we have

$$
\begin{bmatrix}
1 & 2 & -1 \\
0 & -1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2v_1 + 4v_2 - v_3 \\
v_1 - 3v_2 + v_3 \\
v_1 - 2v_2 + v_3
\end{bmatrix}.
$$

The interpretation here is that, for any values of $v_1$, $v_2$, and $v_3$, we can find values $c_1$, $c_2$, and $c_3$ such the $\mathbf{v}$ can be reached as a linear combination of the three vectors gives vectors. In particular, we have

$$
c_1 = 2v_1 + 4v_2 - v_3
$$
$$
c_2 = -v_1 - 3v_2 + v_3
$$
$$
c_3 = -v_1 - 2v_2 + v_3
$$

so that if, for example, we wanted to reach a point $(1, -3, 2)$, we would need $c_1 = -12$, $c_2 = 10$, and $c_3 = 7$. It follows that the span of the vectors is all of $\mathbb{R}^3$. 
Suggested Problems:

1. Determine whether the following set $W$ is a subspace of the indicated space $V$.
   
   (a) $V = \mathbb{R}^3$, $W = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$
   
   (b) $V = \mathbb{R}^3$, $W = \{(s, 2t, -s) \in \mathbb{R}^3 \mid s, t \in \mathbb{R}\}$
   
   (c) $V = \mathbb{R}^2$, $W = \{(t, t^2) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$
   
   (d) $V$ is the set of $2 \times 2$ matrices, $W$ is matrices of form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$ (with standard scalar multiplication and matrix addition operations)

2. Determine whether the given vector $w$ is in the span of the given set of vectors $\{v_1, \ldots, v_m\}$. If it is, give constants $c_i$ such that $w = c_1 \cdot v_1 + \cdots + c_m \cdot v_m$.
   
   (a) $w = (1, -2)$; $v_1 = (-1, 1)$, $v_2 = (2, -2)$
   
   (b) $w = (1, 1, 1)$; $v_1 = (3, 0, 1)$, $v_2 = (1, -2, -1)$
   
   (c) $w = (0, -1, -2)$; $v_1 = (1, -1, -1)$, $v_2 = (1, -2, 2)$, $v_3 = (0, 1, -3)$
   
   (d) $w = (3, 6, 0, 0, 5)$; $v_1 = (1, 0, 2, 1)$, $v_2 = (1, 3, -1, -1, 2)$

3. Determine whether the indicated row $(\ast)$ may be obtained via some combination of elementary row operations for the following linear systems:
   
   (a) $(\ast)$ $\begin{cases} 3x + 2y = 0 \\ x + 3y = -1 \\ 2x - y = 1 \end{cases}$
   
   (b) $(\ast)$ $\begin{cases} y = 0 \\ x - y = 3 \\ 2x + y = -4 \end{cases}$
   
   (c) $(\ast)$ $\begin{cases} 2x + z = -1 \\ y + 2z = -1 \\ -2x + y + z = 0 \end{cases}$
   
   (d) $(\ast)$ $\begin{cases} x - 4y + z = 0 \\ 2x - z = 3 \\ -x + 2y + z = -1 \end{cases}$