Math 133A, Week 7:
Nonhomogeneous Systems, Forced Pendulum

Section 1: Nonhomogeneous Systems

Before returning to consideration of the pendulum model, consider the following second-order differential equation

\[ y'' + 4y = 12x. \quad (1) \]

The only difference between this and the type of equations we have been considering so far is the term 12x on the right-hand side. This extra term makes the differential equation nonhomogeneous \((g(x) \neq 0 \text{ in the standard form})\).

Since the equation is still linear and constant coefficient, we might think that the technique used to solve homogeneous equations might work. Guessing that the solution has the exponential form \(y(x) = e^{rx}\), however, gives

\[ y'' + 4y = e^{rx} (r^2 + 4) = 12x. \]

There is no value of \(r\) which satisfies this equation for all \(x\). The guess we used for homogeneous linear equations will not work for nonhomogeneous equations.

As unsatisfying as it is, we are simply going to guess something else. In particular, we are going to guess a function which, when substituted in the left-hand side, gives the non-homogeneous term 12x on the right-hand side. This turns out to be easier than we probably suspect. We can see very quickly that \(y(x) = 3x\) works since we have \(y''(x) = 0\) so that

\[ y'' + 4y = 4(3x) + 4(0) = 12x. \]

Probably the only way we would not have seen this would have been to overthink the problem!
We will say that we have found a particular solution \( y_p(x) = 3x \) since it satisfies the differential equation and has no undetermined constants. We might wonder, however, how to build a general solution to (1) out of this observation. It is not as easy as multiplying the found solution by a constant, since \( y(x) = Cx \) is not a solution for all \( C \in \mathbb{R} \). For example, we can easily check that \( y(x) = 4x \) is not a particular solution of the equation.

Instead, consider the homogeneous equation

\[
y'' + 4y = 0.
\]

This has the solution \( y_c(x) = C_1 \cos(2x) + C_2 \sin(2x) \). At first this seems unimportant, since (2) and (1) are not the same differential equation, but consider adding (2) to the particular solution \( 3x \) of (1). That is, consider

\[
y(x) = C_1 \cos(2x) + C_2 \sin(2x) + 3x.
\]

We can directly check that is also a solution of (1)! We have

\[
y''(x) = -4C_1 \cos(2x) - 4C_2 \sin(2x)
\]

so that

\[
y'' + 4y = [-4C_1 \cos(2x) - 4C_2 \sin(2x)] + 4[C_1 \cos(2x) + C_2 \sin(2x) + 3x]
\]

\[
= 12x.
\]

We have actually stumbled upon the general solution method for solving second-order (and higher) linear differential equations. The trick will be to find two parts of the solution: one which equals zero when substituted into the LHS, and one which equals the term \( g(x) \).

**Theorem 1**

Any solution of a differential equation of the form

\[
y''(x) + p(x)y'(x) + q(x)y(x) = g(x).
\]

can be written

\[
y(x) = y_c(x) + y_p(x)
\]

where \( y_p(x) \) is any particular solution of (8) and the complementary function \( y_c(x) = C_1y_1(x) + C_2y_2(x) \) is the general solution of the
homogeneous system

\[ y''(x) + p(x)y'(x) + q(x)y(x) = 0. \quad (4) \]

**Proof**

We will prove that a function \( y(x) = yc(x) + yp(x) \) is a solution of (8). We will omit the proof that this is the only form of such a solution.

Suppose that \( yp(x) \) is a particular solution of (8) and \( yc(x) = C_1y_1(x) + C_2y_2(x) \) is a general solution of (4). That is, suppose that

\[
\begin{align*}
yp''(x) + p(x)yp'(x) + q(x)yp(x) &= g(x) \\
yc''(x) + p(x)yc'(x) + q(x)yc(x) &= 0.
\end{align*}
\]

It follows that we have

\[
\begin{align*}
y''(x) + p(x)y'(x) + q(x)y(x) &= \left[y_p''(x) + p(x)y_p'(x) + q(x)y_p(x)\right] \\
&\quad + \left[y_c''(x) + p(x)y_c'(x) + q(x)y_c(x)\right] \\
&= g(x).
\end{align*}
\]

It follows that \( y(x) = yc(x) + yp(x) \) is a solution of (8) and we are done.

**Note:** Since we already know how to solve homogeneous second-order linear DEs with constant coefficients, this result tells us that we need only worry about finding \( yp(x) \). In general, however, it may be very difficult to determine the complementary solution \( yc(x) \) if the coefficients are allowed to vary with \( x \).

**Example 1**

Verify that \( yp(x) = e^{2x} \) is a particular solution of

\[
y'' - 4y' + 5y = e^{2x} \quad (5)
\]
and then find the solution of the corresponding initial value problem with $y(0) = 0$ and $y'(0) = 2$.

**Solution:** To check whether $y_p(x) = e^{2x}$ is a particular solution, we simply need to substitute it into (5). We have

$$y''_p - 4y'_p + 5y_p = (4e^{2x}) - 4(2e^{2x}) + 5(e^{2x}) = e^{2x}.$$  

It follows that $y_p(x) = e^{2x}$ is a particular solution of (5).

In order to solve the IVP, we first need to find the general solution. We know from Theorem 1 that every solution of an equation of the form (5) is of the form $y(x) = y_c(x) + y_p(x)$. It remains only to find the complementary solution $y_c(x)$ to the corresponding homogeneous differential equation

$$y'' - 4y' + 5y = 0.$$  

We guess the solution $y_c(x) = e^{rx}$ to obtain

$$e^{rx}(r^2 - 4r + 5) = 0 \implies r^2 - 4r + 5 = 0.$$  

It follows that we need

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4(5)(1)}}{2} = 2 \pm i.$$  

We therefore have that

$$y_c(x) = C_1 e^{2x} \cos(x) + C_2 e^{2x} \sin(x)$$  

and the corresponding general solution

$$y(x) = C_1 e^{2x} \cos(x) + C_2 e^{2x} \sin(x) + e^{2x}.$$  

In order to solve the initial value problem, we must use $y(0) = 0$ and $y'(0) = 2$ to solve for $C_1$ and $C_2$. We have

$$y'(x) = 2C_1 e^{2x} \cos(x) - C_1 e^{2x} \sin(x) + 2C_2 e^{2x} \sin(x) + C_2 e^{2x} \cos(x) + 2e^{2x}$$
so that
\[ y(0) = 0 = C_1 + 1 \]
\[ y'(0) = 2 = 2C_1 + C_2 + 2. \]

It follows from the first equation that \( C_1 = -1 \). Substituting in the second equation gives \( C_2 = 2 \) so that the desired solution is
\[
y(x) = -e^{2x} \cos(x) + 2e^{2x} \sin(x) + e^{2x}
= e^{2x} (1 + 2 \sin(x) - \cos(x)).
\]

Section 2: Forced Pendulum

A primary application of nonhomogeneous differential equations like (8) is when dealing with a forced pendulum model (also, forced spring). In this situation, we imagine that a pendulum is not only subject to internal restoring and damping forces, but also an external time-dependent force. For instance, we could imagine a steady wind blowing across the pendulum, or coupling the pendulum with a larger system, or simply grabbing the pendulum and shaking it.

In any case, we consider an equation of the form
\[
mx'' + cx' + kx = g(t) \tag{6}
\]
where \( g(t) \) is the external forcing function. This is exactly the form we have just considered so that we know from Theorem 1 that the solution has the form
\[
x(t) = x_c(t) + x_p(t)
\]
where \( x_p(t) \) is a particular solution of (6) and \( x_c(t) \) is a solution of the homogeneous system
\[
mx'' + cx' + kx = 0.
\]

Consider the following example.

Example 2

Consider a pendulum with a mass of 3kg, subject to a restoring force of 4N/m, a frictional force of 8N/(m/s), and sinusoidal forcing \( 65 \sin(t) \)N.
Show that that $x_p(t) = \sin(t) - 8 \cos(t)$ is a solution of the corresponding governing differential equation, and then find the general solution. Describe the transient (i.e. short-term) and long-term behavior of the solution.

**Solution:** We have the governing system of equations

$$3x'' + 8x' + 4x = 65 \sin(t). \tag{7}$$

We have already see that the complementary solution $x_c(t)$ to the complementary problem

$$3x'' + 8x' + 4x = 0$$

is given by $x_c(t) = C_1 e^{-2t} + C_2 e^{-2/3t}$. It remains only to verify the particular solution and use this to construct our general solution $x(t) = x_c(t) + x_p(t)$. We can see that, for $x_p(t) = \sin(t) - 8 \cos(t)$ we have $x_p'(t) = \cos(t) + 8 \sin(t)$ and $x_p''(t) = -\sin(t) + 8 \cos(t)$ so that

$$3x_p'' + 8x_p' + 4x_p$$

$$= 3 [-\sin(t) + 8 \cos(t)] + 8 [\cos(t) + 8 \sin(t)] + 4 [\sin(t) - 8 \cos(t)]$$

$$= 65 \sin(t).$$

It follows that we have the general solution

$$x(t) = x_c(t) + x_p(t) = C_1 e^{-2t} + C_2 e^{-2/3t} + \sin(t) - 8 \cos(t).$$

Even though the constants $C_1$ and $C_2$ depend upon the initial conditions, we can still get a very clear sense of the behavior of the solution. The behavior of the underlying unforced system is given by

$$x_c(t) = C_1 e^{-2t} + C_2 e^{-2/3t}.$$

This corresponds to the fact that the underlying system is **overdamped**. We can consider this portion of the solution as a **transient portion** since it converges to zero. That is, when determining the long-term behavior of the full system (7), the effect of the underlying unforced diminishes over time.
The remaining term

\[ x_p(t) = \sin(t) - 8 \cos(t) \]

correspond to the effect of the external forcing. We can see that this effect is to oscillate indefinitely. This makes sense since we are applying sinusoidal forcing for the system. This solution tells us that the solution will wobble in the short-term, undergo short-term correction from the effect of the underlying system, and then settle into a rhythm which is in phase with the forcing:

Section 3: Undetermined Coefficients

The question now becomes how we find the particular solution \( y_p(x) \). There are two primary methods for such a problem. The first is called the method of undetermined coefficients, and will be an intuitive extension of what we have been doing to date—that is, we will be guessing the solution form.

The second method is called variation of parameters and does not require any guessing. It will, however, require a significantly greater amount of work and some potentially tricky integration—even when we could simply guess the solution. We will largely omit studying variation of parameters.

Consider a general differential equation of the form

\[ ay'' + by' + cy = g(x) \]

(8)

where we want to determine the particular solution \( y_p(x) \). Notice that the LHS of (8) involves just \( y \) and its derivatives, while the RHS contains a known functions of \( x \). What we need is a form of \( y_p(x) \) which can be differ-
entiated to give a function of the form of $g(x)$. Notice that

\[
\frac{d}{dx}[\text{polynomial}] = \text{polynomial}
\]
\[
\frac{d}{dx}[\text{exponential}] = \text{exponential}
\]
\[
\frac{d}{dx}[\text{sine and/or cosine}] = \text{sine and/or cosine}.
\]

This suggests that, if $g(x)$ is a polynomial we should have a \textbf{polynomial} $y_p(x)$, if $g(x)$ is exponential we should have an \textbf{exponential} $y_p(x)$, and if $g(x)$ is trigonometric we should have a \textbf{trigonometric} $y_p(x)$. This suggests the following steps for solving a differential equation of the form (8):

\begin{algorithm}
1. Find the general solution $y_c(x) = C_1y_1(x) + C_2y_2(x)$ of the homogeneous equation
\[
ay''_c + by'_c + cy_c = 0.
\]
2. Select a \textbf{trial function} $y_p(x)$ according to the following:
   (a) If $g(x)$ contains $x^n$ then use
   \[
y_p(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0.
   \]
   (b) If $g(x)$ contains $e^{rx}$ then use
   \[
y_p(x) = Be^{rx}.
   \]
   (c) If $g(x)$ contains $\sin(\alpha x)$ or $\cos(\alpha x)$ then use
   \[
y_p(x) = A \cos(\alpha x) + B \sin(\alpha x).
   \]
3. Substitute the trial function $y_p(x)$ into
   \[
   ay''_p + by'_p + cy_p = g(x)
   \]
   and solve for the undetermined coefficients in $y_p(x)$ (i.e. solve for
   $A, B, A_0, A_1, \ldots$)
\end{algorithm}
4. If relevant, use the initial conditions to solve for the undetermined constants in the general solution $y(x) = y_c(x) + y_p(x)$.

Note: We may need to use combinations of these functions. For example, if we have $g(x) = e^x \sin(x)$, we need to use $y_p(x) = Ae^x \cos(x) + Be^x \sin(x)$. If we have $g(x) = x^2 e^{-x}$, we would need to use $y_p(x) = (Ax^2 + Bx + C)e^{-x}$, etc. Also note that the arguments inside the trigonometric and exponential terms are also important. For instance, the forcing term $g(x) = \sin(x) + \cos(2x)$ requires the trial function $y_p(x) = A \cos(x) + B \sin(x) + C \cos(2x) + D \sin(2x)$ while $g(x) = \sin(x) + \cos(x)$ requires only $y_p(x) = A \sin(x) + B \cos(x)$.

Example 3

Find the general solution of the differential equation

$$y'' + 4y = e^{-x} - 3x^3.$$ 

Solution: We need to first solve the homogeneous equation

$$y'' + 4y = 0.$$ 

The guess $y_c(x) = e^{rx}$ gives $e^{rx}(r^2 + 4) = 0$ so that $r = \pm 2i$. It follows that

$$y_c(x) = C_1 \cos(2x) + C_2 \sin(2x).$$

We now need to use a trial function $y_p(x)$ with a suitable form that it could give $e^{-x} - 3x^3$ after differentiation. We try

$$y_p(x) = Ae^{-x} + Bx^3 + Cx^2 + Dx + E$$

$$\Rightarrow y'_p(x) = -Ae^{-x} + 3Bx^2 + 2Cx + D$$

$$\Rightarrow y''_p(x) = Ae^{-x} + 6Bx + 2C.$$
It follows that the differential equation gives
\begin{align*}
y_p''(x) + 4y_p(x) \\
&= (Ae^{-x} + 6Bx + 2C) + 4(Ae^{-x} + Bx^3 + Cx^2 + Dx + E) \\
&= 5Ae^{-x} + 4Bx^3 + 4Cx^2 + (6B + 4D)x + (2C + 4E) \\
&= e^{-x} - 3x^3.
\end{align*}

It follows that we need to satisfy
\begin{align*}
5A &= 1 \\
4B &= -3 \\
4C &= 0 \\
6B + 4D &= 0 \\
2C + 4E &= 0.
\end{align*}

It follows that we have $A = 1/5$, $B = -3/4$, $C = 0$, $D = 9/8$, and $E = 0$. The corresponding particular solution is
\begin{equation*}
y_p(x) = \frac{1}{5} e^{-x} - \frac{3}{4} x^3 + \frac{9}{8} x.
\end{equation*}

The general solution is therefore
\begin{equation*}
y(x) = y_c(x) + y_p(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{5} e^{-x} - \frac{3}{4} x^3 + \frac{9}{8} x.
\end{equation*}

Example 4

A spring with a mass of 1 kg is stretched 1 m by a force of 2 N and experiences damping of 3 N when the velocity is 1 m/s. Supposing there is an external force of $g(t) = 2e^{-t} \sin(t)$ N and the spring is released from rest 1m from its resting position, determine the solution of the system.

Solution: The given information tells us that we have $m = 1$ kg, $c = 3$ N/(m/s), and $k = 2$ N/m, so that
\begin{equation*}
x'' + 3x' + 2x = 2e^{-t} \sin(t).
\end{equation*}

We first consider the homogeneous system $x'' + 3x' + 2x = 0$. Guessing
the solution \( x(t) = e^{rt} \) gives \( r^2 + 3r + 2 = (r + 2)(r + 1) = 0 \) so that \( r = -1 \) and \( r = -2 \). It follows that
\[
x_c(t) = C_1 e^{-t} + C_2 e^{-2t}.
\]
To find the particular solution \( x_p(t) \), we guess the trial form
\[
x_p(t) = A e^{-t} \sin(x) + B e^{-t} \cos(x).
\]
This gives
\[
\begin{align*}
x'_p(t) &= -(A + B) e^{-t} \sin(x) + (A - B) e^{-t} \cos(x) \\
x''_p(t) &= 2B e^{-t} \sin(t) - 2A e^{-t} \cos(t).
\end{align*}
\]
It follows that we have
\[
x''_p + 3x'_p + 2x_p = [2B e^{-t} \sin(t) - 2A e^{-t} \cos(t)]
+ 3 [-(A + B) e^{-t} \sin(x) + (A - B) e^{-t} \cos(x)]
+ 2 [A e^{-t} \sin(x) + B e^{-t} \cos(x)]
= -(A + B) e^{-t} \sin(t) + (A - B) e^{-t} \cos(t)
= 2e^{-t} \sin(t).
\]
It follows that we have
\[
\begin{align*}
-A - B &= 2 \\
A - B &= 0.
\end{align*}
\]
so that, from the second equation, \( A = B \), which implies \( A = B = -1 \) by the first equation. It follows that we have the general solution
\[
x(t) = C_1 e^{-t} + C_2 e^{-2t} - e^{-t} \sin(t) - e^{-t} \cos(t).
\]
We have the initial conditions \( x(0) = 1 \) and \( x'(0) = 0 \). We have
\[
x'(t) = -C_1 e^{-t} - 2C_2 e^{-2t} + 2e^{-t} \sin(t)
\]
so that
\[
\begin{align*}
x(0) &= 1 = C_1 + C_2 - 1 \\
x'(0) &= 0 = -C_1 - 2C_2.
\end{align*}
\]
It follows from the second equation that $C_1 = -2C_2$ so that $C_2 = -2$ from the first. It follows that $C_1 = 4$ so that our particular solution is

$$x(t) = 4e^{-t} - 2e^{-2t} - e^{-t} \sin(t) - e^{-t} \cos(t).$$

Section 4: Modified Trial Forms

It turns out that we have only partially answered the question of how to solve a general second-order system with constant coefficients. Consider finding the general solution of the differential equation

$$y'' + 4y = \cos(2x) \quad (9)$$

with the method given by Algorithm 1.

The complementary function is easily determined to be $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$. We confidently guess the trial function

$$y_p(x) = A \cos(2x) + B \sin(2x).$$

This gives

$$y_p'(x) = -2A \sin(2x) + 2B \cos(2x)$$

$$y_p''(x) = -4A \cos(2x) + 4B \sin(2x).$$

However, we can easily check that

$$y_p''(x) + 4y_p(x) = -4A \cos(2x) + 4B \sin(2x) + 4(A \cos(2x) + B \sin(2x)) = 0.$$ 

We need to match constants so that this equals $g(x) = \sin(2x)$ but the terms on the LHS have vanished. There are no constants left to solve for!
Something has gone seriously wrong, but after a moment of thought we realize that we should have expected this. The complementary function is
\[ y_c(x) = C_1 \cos(2x) + C_2 \sin(2x) \] so that the combination of functions in the given trial function had to vanish when it was substituted into the LHS of (9). Algorithm 1 will never work for differential equations where the complementary function \( y_c(x) \) contains terms requires of the proposed trial form \( y_p(x) \)!

For such differential equations, the fix is to choose a different trial function. We have the following modification of step 2. of Algorithm 1.

Algorithm 2

2.* If the complementary solution \( y_c(x) \) contains common terms with the required trial functions \( y_p(x) \) given in step 2. of Algorithm 1, then select the corresponding portion of the trial function \( y_p(x) \) according to the following:

(a*) \( y_p(x) = A_n x^{n+s} + A_{n-1} x^{n+s-1} + \cdots + A_1 x^{s+1} + A_0 x^s \)

(b*) \( y_p(x) = B x^s e^{rx} \)

(c*) \( y_p(x) = A x^s \cos(ax) + B x^s \sin(ax) \)

where \( s \) is the lowest power which produces a term which is independent of those in \( y_c(x) \).

Note: We will not offer a rigorous proofs of the forms in Algorithm 2 in this course. It should be noted that the notion of multiplying by the independent variable \( x \) to generate independent solutions is a common technique, and was used previously to generate the solutions \( y(x) = C_1 e^{rx} + C_2 x e^{rx} \) for differential equations with repeated roots.

Example 5

Find the general solution of

\[ y'' + 4y = \cos(2x). \]
Solution: The complementary function was $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$ so we are not allowed to use $y_p(x) = A \cos(2x) + B \sin(2x)$ as a trial function. Instead, we must use

$$y_p(x) = Ax \cos(2x) + Bx \sin(2x).$$

This gives

$$y'_p(x) = A \cos(2x) + B \sin(2x) - 2Ax \sin(2x) + 2Bx \cos(2x)$$

$$y''_p(x) = 4B \cos(2x) - 4A \sin(2x) - 4Ax \cos(2x) - 4Bx \sin(2x).$$

Plugging into the DE gives

$$y''_p + 4y_p$$

$$= 4B \cos(2x) - 4A \sin(2x) - 4Ax \cos(2x) - 4Bx \sin(2x) + 4(Ax \cos(2x) + Bx \sin(2x))$$

$$= 4B \cos(2x) - 4A \sin(2x)$$

$$= \cos(2x).$$

It follows that we need $A = 0$ and $B = 1/4$ so that we have the particular solution

$$y_p(x) = \frac{1}{4} x \sin(2x).$$

The general solution of the differential equation is therefore

$$y(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{4} x \sin(2x).$$

Example 6

Find the general solution of

$$y'' - 2y' + y = 2e^x + 4e^{-x}.$$ 

Solution: We first solve the complementary problem

$$y''_c - 2y'_c + y_c = 0.$$
We guess the solution $y_c(x) = e^{rx}$ to get
\[ e^{rx}(r^2 - 2r + 1) = e^{rx}(r - 1)^2 = 0 \]
so that we have the repeated root $r = 1$. It follows that the complementary solution is
\[ y_c(x) = C_1e^x + C_2xe^x. \]
We now consider trial forms for the particular solution. Notice, however, that the natural trial form
\[ y_p(x) = Ae^x + Be^{-x} \]
overlaps with the complementary solution $y_c(x)$ in the term $e^x$. This choice of $y_p(x)$ therefore does not have enough degrees of freedom to solve for all of the undetermined constants. We attempt instead to multiply this problematic portion of this trial form by $x$. (Notice that we do not need to multiply the term $e^{-x}$ by $x$ since we have enough degrees of freedom to solve for $B$.) This gives
\[ y_p^*(x) = Axe^x + Be^{-x}. \]
We notice, however, that this form still overlaps with $y_c(x)$ (in the term $xe^x$ this time). It follows that we need to multiply by $x$ again to get
\[ y_p^{**}(x) = Ax^2e^x + Be^{-x}. \]
It is only now, after multiplying by $x$ twice, that we can check that the trial form $y_p(x)$ does not contain any overlap with $y_c(x)$.

We now plug $y_p^{**}(x)$ (which we relabel $y_p(x)$), into the differential equation. We have
\[ y_p'(x) = Ax^2e^x + 2Axe^x - Be^{-x} \]
\[ y_p''(x) = Ax^2e^x + 4Axe^x + 2Ae^x + Be^{-x}. \]
Substituting in the differential equation gives:

\[ y_p'' - 5y_p' + 6y_p = [Ax^2e^x + 4Axe^x + 2Ae^x + Be^{-x}] \]
\[ - 2[Ax^2e^x + 2Axe^x - Be^{-x}] \]
\[ + [Ax^2e^x + Be^{-x}] \]
\[ = 2Ae^x + 4Be^{-x}. \]

Since the RHS of the differential equation was \(2e^x + 4e^{-x}\), we can easily see that we need \(A = 1\) and \(B = 1\). It follows that the particular solution is

\[ y_p(x) = x^2e^x + e^{-x}. \]

The general solution is therefore

\[ y(x) = y_c(x) + y_p(x) = C_1e^x + C_2xe^x + x^2e^x + e^{-x}. \]

### Suggested Problems

1. Show that the following differential equations have the given particular solutions \(y_p(x)\). Then determine general solution, and solution to the given initial value problem. (All derivatives with respect to \(x\).)

   (a) \(y'' + 2y' + y = x\)
   \(y_p(x) = x - 2\)
   \(y(0) = 1, \ y'(0) = 0\)

   (b) \(y'' - 4y = x^2 - \frac{1}{2}\)
   \(y_p(x) = -\frac{1}{4}x^2\)
   \(y(0) = 2, \ y'(0) = 0\)

   (c) \(y'' + 4y = 3\sin(x)\)
   \(y_p(x) = \sin(x)\)
   \(y(0) = -1, \ y'(0) = 1\)

   (d) \(y'' + 6y' + 10y = 5e^{-x} + 10e^{4x}\)
   \(y_p(x) = e^{-x} + e^{4x}\)
   \(y(0) = 0, \ y'(0) = 1\)

2. Use Algorithm 1 to solve the following initial value problems (all derivatives with respect to \(x\)):

   (a) \[ \begin{cases} 
   y'' - 3y' + 2y = 2e^{3x}, \\
   y(0) = -1, \\
   y'(0) = 0 
   \end{cases} \]

   (b) \[ \begin{cases} 
   2y'' + 5y' - 3y = 10\sin(x), \\
   y(0) = 1, \\
   y'(0) = 0 
   \end{cases} \]
3. Use Algorithm 1 with the modified step 2* to solve the following initial value problems (all derivatives with respect to $x$):

(a) \[
\begin{cases}
y'' + 3y' + 2y = e^x, \\
y(0) = 0, \\
y'(0) = 0
\end{cases}
\]

(b) \[
\begin{cases}
4y'' + 9y = 12\sin\left(\frac{3}{2}x\right), \\
y(0) = 3, \\
y'(0) = -1
\end{cases}
\]

(c) \[
\begin{cases}
y'' - 6y' + 9y = 6xe^{3x}, \\
y(0) = 1, \\
y'(0) = -1
\end{cases}
\]

(d) \[
\begin{cases}
y'' + 4y' + 5y = e^{-2x}\sin(x), \\
y(0) = 0, \\
y'(0) = 0
\end{cases}
\]