Instructions

1. Fill out this cover page completely.

2. Answer questions in the space provided, using scratch paper for rough work.

3. Show all the work required to obtain your answers.

4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

FOR EXAMINERS’ USE ONLY

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1. Short Answers and Proofs:

(a) The second-order linear differential equation $y'' + p(x)y' + q(x)y = g(x)$ is **homogeneous** if...

(b) Suppose that $y_1(x)$ and $y_2(x)$ are solutions of $ay'' + by' + cy = 0$. Prove that $y(x) = C_1y_1(x) + C_2y_2(x)$ is also a solution for any $C_1, C_2 \in \mathbb{R}$.

2. True/False:

(a) The function $y(x) = \frac{1}{x}$ is a solution of $x^2y'' + 3xy' + y = 0$. [True / False]

(b) The unforced linear pendulum model $mx'' + cx' + kx = 0$ exhibits oscillations for all values of $m, c, k > 0$. [True / False]

(c) Consider the undamped, forced pendulum model $x'' + x = \cos(\omega t)$. The maximal amplitude of oscillations in the solution $x(t)$ does not depend upon the choice of $\omega$. [True / False]
3. **Complementary and Particular Solutions:**

Consider the following second order differential equation (derivatives with respect to $x$):

$$4y'' - 5y' + y = g(x)$$

(1)

(a) Determine the complementary solution $y_c(x)$ of (1).

(b) Determine the trial function form $y_p(x)$ for (1) given the following $g(x)$ (you do not need to solve for the constants!):

(i) $g(x) = \frac{1}{4}x \sin(2x)$

(ii) $g(x) = xe^x$

(c) Determine the particular solution $y_p(x)$ for (1) for the following $g(x)$ (now you have to solve for the constants!): $g(x) = 10e^{-x} + x$
4. Applications: Damped Pendulum

Consider a 1 kg mass attached to a pendulum which has a restoring constant of 5 N/m and a damping constant of 2 N/(m/s). Suppose the pendulum is initially displaced to the right 1 meter and given an initial push of 3 m/s to the left. This gives rise to the linear pendulum model:

\[
\begin{align*}
    x'' + 2x' + 5x &= 0, \\
    x(0) &= 1, \\
    x'(0) &= -3
\end{align*}
\]

(2)

(a) Classify the behavior of the pendulum modeled by (2) as undamped, underdamped, critically damped, or overdamped.

(b) Determine the solution \( x(t) \) of the initial value problem (2).

(c) Describe the long-term behavior of \( x(t) \) as \( t \to \infty \). Interpret this in terms of the behavior of the physical pendulum over time.
5. Applications: Resonance

Suppose we incorporate an oscillatory external force of $17 \cos(\omega t)$ N in the pendulum model (2). Neglecting initial conditions, this gives rise to the **forced linear pendulum model**:

$$x'' + 2x' + 5x = 17 \cos(\omega t).$$  \hspace{1cm} (3)

(a) Determine the particular solution $x_p(t)$ of (3) when $\omega = 2$ (the natural frequency of the unforced system).  

(c) For general $\omega$, the particular solution $x_p(t)$ can be written in the form $A(\omega) \cos(2t - \alpha)$ with the amplitude function

$$A(\omega) = \frac{17}{\sqrt{\omega^4 - 6\omega^2 + 25}}.$$  

Determine the frequency $\omega > 0$ which yields the maximal amplitude. How does this frequency compare with the natural frequency of the unforced system ($\omega = 2$)?
6. Linear Algebra

Consider the following matrix:

\[ A = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix} \]

(a) Determine the inverse \( A^{-1} \) of \( A \).

(b) Determine the eigenvalues, eigenvectors, and, if necessary, generalized eigenvectors of \( A \).