Math 123, Spring 2016, Term Test II
Differential Equations and Linear Algebra
(version 1)

Date: Wednesday, April 20
Time: 1:30-2:45 p.m.
Instructor: Matthew Johnston

Last Name: __________________________
First Name: _________________________
SJSU Student ID Number: __________________________

Instructions
1. Fill out this cover page completely.
2. Answer questions in the space provided, using scratch paper for rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

FOR EXAMINERS’ USE ONLY

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1. Short Answer:

(a) Give an example of a non-trivial separable differential equation. [As a minimum, let’s say that both $x$ and $y$ must appear somewhere in the equation.]

(b) A first-order differential equation is power-homogeneous if it can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

Show that the transformation $v = \frac{y}{x}$ takes a power-homogeneous differential equation into the separable differential equation

$$\frac{dv}{dx} = \frac{F(v) - v}{x}.$$

2. True/False:

(a) In general, a differential equation $y' = f(x, y)$ may have a family of solutions. Specifying an initial condition $y(x_0) = y_0$ selects a specific solution (or solutions) from the family.

[True / False]

(b) The integrating factor for $y' + \frac{x}{x^2 + 1}y = x^2$ is $\mu(x) = \sqrt{x^2 + 1}$.

[True / False]

(c) The function $y(x) = \frac{1}{x^2}$ is a solution of $x^2y'' + xy' - y = 0$.

[True / False]
3. Slope Fields and Solutions:

Consider the following initial value problem:

\[
\begin{aligned}
\frac{dy}{dx} &= \sin(x) - y \\
y(0) &= -\frac{1}{2}
\end{aligned}
\]  \hspace{1cm} (1)

(a) Sketch the slope field of (1) in the \((x, y)\)-plane.  
[Hint: Setting \(y' = C\) gives \(y = \sin(x) - C\).]

(b) Show that \(y(x) = \frac{1}{2} (\sin(x) - \cos(x))\) is a solution of the initial value problem (1).  
[Hint: Remember to check the initial condition!]

(c) Describe the behavior of the solution as \(x \to \infty\). In particular, does the solution converge to a specific value, or approach some sort of identifiable behavior?
4. First-Order Differential Equations:

Solve the following first-order differential equations:

\[
\begin{align*}
\text{(a)} & \quad \frac{dy}{dx} - \frac{1}{x} y = 3xy^2, \\
& \quad y(1) = 1
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad \frac{dy}{dx} = \frac{y + \sqrt{xy}}{x}, \quad x > 0
\end{align*}
\]
5. Applications: (Inflow/Outflow)

Consider a mixing tank with a volume of 25 gallons which is initially filled with 5 gallons of fresh water. Suppose there is an inflow pipe which pumps in a 1 lb/gallon brine (salt/water) mixture at a rate of 3 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallons per minute.

(a) Use the given information to derive an initial value problem which models the amount of salt in the tank as a function of time. [Note: Do not forget the initial condition!]

(b) Find the solution of the initial value problem derived in part (a).

(c) How much salt is in the tank when the tank is full? [Note: You do not need to evaluate to a decimal value, although, if it helps, \( \frac{125}{625} = 0.2 \).]
6. Second-Order Differential Equations

Solve the following second-order differential equations (all derivatives with respect to $x$):

- (a) $2y'' + y' - 3y = 0$

- (b) \[
\begin{aligned}
\quad & y'' + 4y' + 4y = 0 \\
\quad & y(0) = -1 \\
\quad & y'(0) = 2
\end{aligned}
\]
THIS PAGE IS FOR ROUGH WORK