Math 123, Spring 2016, Term Test II  
Differential Equations and Linear Algebra  
(version 2)

**Date:**  Wednesday, April 20  
**Time:**  1:30-2:45 p.m.  
**Instructor:**  Matthew Johnston

Last Name:  
First Name:  
SJSU Student ID Number:  

**Instructions**

1. Fill out this cover page completely.
2. Answer questions in the space provided, using scratch paper for rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted but you may consult a one page hand-written cheat sheet.

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1. Short Answer:

(a) Give an example of a non-trivial first-order linear differential equation. [As a minimum, let's say that both \( y \) and \( y' \) must appear somewhere in the equation.]

Anything of form \( y' + p(x)y = q(x) \)

\( \begin{align*}
\text{e.g., } & y' + \frac{1}{x}y = e^x \quad y' + 5\ln(x)y = x^2 + \ln(x) \\
y' + y = 7 \\
& e + c...
\end{align*} \)

(b) A first-order differential equation is power-homogeneous if it can be written in the form

\[
\frac{dy}{dx} = F\left(\frac{y}{x}\right).
\]

Show that the transformation \( v = \frac{y}{x} \) takes a power-homogeneous differential equation into the separable differential equation

\[
\frac{dv}{dx} = \frac{F(v) - v}{x}.
\]

\[
\begin{align*}
\forall = \frac{y}{x} & \Rightarrow v + xv' = F(v) \\
\Rightarrow y = xv & \Rightarrow xv' = F(v) - v \\
\Rightarrow y' = v + xv' & \Rightarrow v' = \frac{F(v) - v}{x}
\end{align*}
\]

2. True/False:

(a) In general, a differential equation \( y' = f(x, y) \) may have a family of solutions. Specifying an initial condition \( y(x_0) = y_0 \) selects a specific solution (or solutions) from the family.

(True) / False

(b) The integrating factor for \( y' + \frac{2x}{x^2 + 1}y = \frac{1}{x} \) is \( \mu(x) = \sqrt{x^2 + 1} \).

(True) / False

\[
\mu(x) = e^{\int \frac{2x}{x^2 + 1} dx} = e^{\ln(x^2 + 1) = x^2 + 1 \neq \sqrt{x^2 + 1}}
\]

(c) The function \( y(x) = \frac{1}{x} \) is a solution of \( x^2y'' + xy' - y = 0 \).

(True) / False

\[
\begin{align*}
y = \frac{1}{x} & \Rightarrow y' = -\frac{1}{x^2}, \quad y'' = \frac{2}{x^3} \\
\Rightarrow x^2 y'' + xy' - y & = \frac{2}{x} - \frac{1}{x} - \frac{1}{x} = 0 \quad (\checkmark)
\end{align*}
\]
3. Slope Fields and Solutions:

Consider the following initial value problem:
\[
\begin{cases}
\frac{dy}{dx} = \cos(x) - y \\
y(0) = \frac{1}{2}
\end{cases}
\]  
(1)

(a) Sketch the slope field of (1) in the \((x,y)\)-plane.

[Hint: Setting \(y' = C\) gives \(y = \cos(x) - C\).]

(b) Show that \(y(x) = \frac{1}{2} (\sin(x) + \cos(x))\) is a solution of the initial value problem (1).

[Hint: Remember to check the initial condition!]

\[
LHS: \ y' = \frac{1}{2} (\cos(x) - 5\sin(x)) \\
RHS: \ \cos(x) - y = \cos(x) - \frac{1}{2} (\sin(x) + \cos(x)) \\
= \frac{1}{2} (\cos(x) - \sin(x)) = RHS \quad \Box
\]

(c) Describe the behavior of the solution as \(x \to \infty\). In particular, does the solution converge to a specific value, or approach some sort of identifiable behavior?

Solution oscillates indefinitely.
4. First-Order Differential Equations:

Solve the following first-order differential equations:

(a) \[ \frac{dy}{dx} - \frac{1}{x} y = \frac{1}{2y}, \quad y(1) = 1 \]

\[ v = y^{1-n} = y^{1-(-1)} = y^2 \]

\[ \Rightarrow y = v^{\frac{1}{2}} \]

\[ \Rightarrow v^{-\frac{1}{2}} \]

\[ \Rightarrow v^\prime = \frac{1}{2} v^{-\frac{1}{2}} v^1 \]

\[ \Rightarrow \frac{1}{2} v^{\frac{1}{2}} v - \frac{1}{x} v^{\frac{1}{2}} = \frac{1}{2} v^{\frac{1}{2}} \]

\[ \Rightarrow 2 v^{\frac{1}{2}} = v' - \frac{2}{x} v = 1 \]

\[ N(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln(x)} = e^{2 \ln(x)} = \frac{1}{x^2} \]

\[ \frac{dy}{dx} = \frac{2x^2 - y^2}{xy}, \quad x > 0 \]

(b) Power homogeneous:

\[ v = \frac{y}{x} \]

\[ \Rightarrow \int v + xv^1 = \frac{2x^2 - x^2 v^2}{x^2 v} = \frac{2 - v^2}{v} \]

\[ \Rightarrow xv^1 = \frac{2 - v^2}{v} - v = \frac{2 - v^2 - v^2}{v} = 2 \left( \frac{1 - v^2}{v} \right) \]

\[ \Rightarrow \frac{v}{2(1-v^2)} dv = \frac{1}{x} dx \]

\[ \Rightarrow \frac{1}{2} \ln(1-v^2) = \ln(x) + C \]

\[ \Rightarrow \ln(1-v^2) = \ln(x)^2 + C \quad (C = -4C) \]

\[ \Rightarrow 1 - v^2 = e^{k \ln(x)} \]

\[ \Rightarrow v^2 = 1 - \frac{k}{x^2} \]

\[ \Rightarrow v = \pm \sqrt{1 - \frac{k}{x^2}} \Rightarrow y = \pm x \sqrt{1 - \frac{k}{x^4}} = \pm \frac{x}{x} \sqrt{1 - \frac{k}{x^4}} \]
5. Applications: (Inflow/Outflow)

Consider a mixing tank with a volume of 50 gallons which is initially filled with 10 gallons of fresh water. Suppose there is an inflow pipe which pumps in a 1 lb/gallon brine (salt/water) mixture at a rate of 4 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallons per minute.

\[ V(t) = 10 + (4-2)t = 10 + 2t \]

\[ \frac{dA}{dt} = (1)(4) - \frac{A}{10 + 2t} \]

\[ A(0) = 0 \]

\[ A(t) = \int \left(10 + 2t\right) dt = \int \left(10 + 2t\right) dt = \left(10 + 2t\right) + C \]

\[ \Rightarrow A(0) = 0 \Rightarrow 0 = 10 + 2(0) + C \]

\[ C = -100 \]

\[ \Rightarrow A(t) = 10 + 2t - \frac{100}{10 + 2t} \]

Rewritten:

\[ A(t) = \frac{2t^2 + 20t}{5 + t} \]

(b) Find the solution of the initial value problem derived in part (a).

(c) How much salt is in the tank when the tank is full? [Note: You do not need to evaluate to a decimal value, although, if it helps, 1200/25 = 48.]

Full at \( V = 50 \Rightarrow \frac{V}{2} = 25 \)

\[ 2 + \frac{40}{2} \Rightarrow t = 20 \]

\[ A(20) = \left(10 + 2\cdot20\right) - \frac{100}{10 + 2\cdot20} \]

\[ = 50 - \frac{100}{50} \]

\[ = 48 \]
6. Second-Order Differential Equations

Solve the following second-order differential equations (all derivatives with respect to $x$):

\[ 2y'' + 4y' - 6y = 0 \]

(a) \[ y = e^{rx} \Rightarrow e^{rx}(2r^2 + 4r - 6) = 0 \]
\[ = 2e^{rx}(r^2 + 2r - 3) = 0 \]
\[ = 2e^{rx}(r+3)(r-1) = 0 \]
\[ \Rightarrow r = 1, \quad r = -3 \]
\[ \Rightarrow y(x) = C_1 e^x + C_2 e^{-3x} \]

(b) \[ \begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \]
\[ y = e^{rx} \Rightarrow e^{rx}(r^2 + 4r + 5) = 0 \]
\[ r = -4 \pm \sqrt{16 - 4(5)} = -2 \pm \frac{\sqrt{4}}{2} = -2 \pm i \]
\[ \Rightarrow \alpha = -2, \quad \beta = 1 \]
\[ \Rightarrow y(x) = C_1 e^{-2x} \cos x + C_2 e^{-2x} \sin x \]
\[ y'(x) = -2C_1 e^{-2x} \cos x - C_1 e^{-2x} \sin x - 2C_2 e^{-2x} \sin x + C_2 e^{-2x} \cos x \]
\[ \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \Rightarrow \begin{cases} 1 = C_1 \\ 0 = -2C_1 + C_2 \end{cases} \]
\[ \Rightarrow 0 = -2(1) + C_2 \Rightarrow C_2 = 2 \]
\[ \Rightarrow y(x) = e^{-2x} \cos x + 2 e^{-2x} \sin x, \]
THIS PAGE IS FOR ROUGH WORK