Section 1: Separable DEs

We are finally to the point in the course where we can consider how to find solutions to differential equations. We start with a motivating example.

**Example 1**

Consider the first-order ODE

\[ y' = \frac{1 - y}{x}. \]

The only method we have discussed so far for finding solutions \( y(x) \) is direct integration. As written, however, the RHS of this DE depends on both \( x \) and \( y \) and so cannot be integrated directly to yield a solution \( y \). We will need another method of approach.

Our trick will to notice that we can make the problem look like an integration problem with respect to \( x \) by removing the \( y \) from the right-hand side and moving the differential \( dx \) to the other side. This leaves us with

\[ \frac{dy}{1 - y} = \frac{dx}{x}. \]

Not only does the right-hand side look like an integral question (with respect to \( x \)), but the left-hand side looks like an integral question as well (with respect to \( y \)). This is exactly how we will treat the equation.
We integrate (wrt $y$ on the left and $x$ on the right) to obtain

$$
\int \frac{1}{1-y} \, dx = \int \frac{1}{x} \, dx \implies -\ln |1-y| = \ln |x| + C \implies |1-y| = \frac{\tilde{C}}{|x|}
$$

where $\tilde{C} = e^{-C} > 0$. There are a few technical details to sort out regarding the absolute value. The good news is that we will only perform the full analysis once and be able to extend to further examples. We have the following four cases:

- $y > 1, x > 0 \implies -(1-y) = \frac{\tilde{C}}{x} \implies y = 1 + \frac{\tilde{C}}{x}, \tilde{C} > 0$
- $y > 1, x < 0 \implies -(1-y) = -\frac{\tilde{C}}{x} \implies y = 1 - \frac{\tilde{C}}{x}, \tilde{C} > 0$
- $y < 1, x > 0 \implies (1-y) = \frac{\tilde{C}}{x} \implies y = 1 - \frac{\tilde{C}}{x}, \tilde{C} > 0$
- $y < 1, x < 0 \implies (1-y) = -\frac{\tilde{C}}{x} \implies y = 1 + \frac{\tilde{C}}{x}, \tilde{C} > 0$

Recognizing that $y = 1$ (i.e. $\tilde{C} = 0$) is a trivial solution, we have that the sign of $\tilde{C}$ does not actually matter. The general solution is

$$
y = 1 + \frac{\tilde{C}}{x}
$$

where $\tilde{C} \in \mathbb{R}$ (that is, the constant can be $< 0$, $= 0$, or $> 0$, which was not previously allowed since $\tilde{C} = e^{-C} > 0$).

What we have discovered in this example is a class of DEs for which integration is sufficient to determine the solution $y(x)$. We formally introduce the following.

**Definition 1**

The first order DE $y' = f(x, y)$ is called **separable** if it can be written in the form

$$
g(y) \frac{dy}{dx} = h(x) \quad \text{or} \quad g(y) \, dy = h(x) \, dx
$$

for some function $g$ of $y$ and $h$ of $x$. 

2
That is, a DE is separable if the dependence on \( x \) and \( y \) in \( f(x, y) \) may be separated. In such cases, we may then integrate the sides separately and rearrange to obtain a solution \( y(x) \). A rigorous justification of the validity of this separate and integrate method follows from the chain rule but will not be covered here.

**Note:** Integration will factor significant in the analysis of separable equations. Every time we evaluate such a DE, we will have to integrate not just once, but *twice.*

**Example 2**

Solve the IVP

\[
y' = -y^2 \frac{(1 + 2x^2)}{x}, \quad y(1) = 1.
\]

**Solution:** Notice that, if we divide the equation by \( y^2 \) and move the differential \( dx \) to the RHS, we have

\[
\frac{1}{y^2} dy = -\frac{(1 + 2x^2)}{x} dx.
\]

This is perfect! We have isolated the dependence on \( y \) on the LHS, and the dependence on \( x \) on the RHS. Now it is only a matter of integrating (twice!). We have

\[
\int \frac{1}{y^2} dy = - \int \left( \frac{1}{x} + 2x \right) dx
\]

\[\implies -y = - \ln(x) - x^2 + C\]

\[\implies y = \frac{1}{\ln(x) + x^2 - C}.
\]

The initial condition \( y(1) = 1 \) gives

\[
1 = \frac{1}{1 - C}
\]
so that $C = 0$. It follows that the particular solution is

$$y = \frac{1}{\ln(x) + x^2}.$$ 

**Section 2: First-Order Linear DEs**

We now introduce another common solution method for first-order DEs. We start by re-considering the original DE

$$y' = \frac{1 - y}{x}.$$ 

Supposing we were unfamiliar with the method of separation, we might notice that we can rewrite the expression as:

$$xy' + y = 1.$$ \hspace{1cm} (1)

There is nothing in the expression dictating that we would want to represent this equation in this form, but we can at the very least notice one nice thing about this form: it was easy to classify! It is a **first-order linear differential equation**.

There is a little bit of cheating that has been done in rearranging the expression this way, but it is a suggestive bit of cheating. Let’s focus on the LHS of the above expression:

$$xy' + y.$$ 

If we stare this for long enough, or were born with unparalleled mathematical powers, we might notice that this can be written in a more compact form. Without justifying, for a moment, why we would want to do this, we might notice that this expression is the end result of the product rule for differentiation on the term $xy$. We have

$$\frac{d}{dx} [xy] = xy' + y.$$ 

In other words, we can take the two terms on the left-hand side and condense them into a single term, at the expense of having to recall the product rule for differentiation. We can now rewrite (1) as

$$\frac{d}{dx} [xy] = 1.$$
This is a huge improvement over our previous expression. The reason should be clear: we can integrate it! If we integrate the left-hand and right-hand sides by $x$, the Fundamental Theorem of Calculus tells us the differential on the left-hand side disappears, and the right-hand side can be evaluated as long as we know an anti-derivative of whatever the term there happens to be. That is, we have

$$\int \frac{d}{dx} [xy] \, dx = \int 1 \, dx$$

$$\Rightarrow \quad xy = x + C, \quad C \in \mathbb{R}$$

which, after dividing by $x$, implies that we have the general solution

$$y(x) = 1 + \frac{C}{x}, \quad C \in \mathbb{R}.$$ 

This coincides with the solution obtained by separating the variables and integrating directly.

To see whether this method is sufficient to solve all first-order linear DEs, consider the example

$$xy' + 2y = 1.$$  

(2)

This is only subtly different that the previous example—in fact, the only difference is the coefficient of the $y$ term is now two. This subtle difference, however, is enough to sabotage our earlier intuition with regards to a solution method, since there is no function $f(x)$ such that

$$\frac{d}{dx} [f(x) \, y] = xy' + 2y.$$ 

So what can we do?

Let’s consider changing the expression (again!) but in a different way. Let’s multiply (2) by a single term that is a function of $x$. In this case, we will choose the function to be $x$ itself. This gives us

$$x^2 y' + 2xy = x.$$ 

If there were any questions with regards to why we would want to do that, they have now been answered. We have that

$$\frac{d}{dx} [x^2 y] = x^2 y' + 2xy = x.$$ 

Again, we can integrate to get the solution. We have

$$\int \frac{d}{dx} [x^2 y] \, dx = \int x \, dx$$
⇒ \[ x^2y = \frac{x^2}{2} + C, \quad C \in \mathbb{R} \]

so that the desired solution is

\[ y(x) = \frac{1}{2} + \frac{C}{x^2}, \quad C \in \mathbb{R}. \]

The difference with this example was that we had to multiply by some factor before we could use the product rule trick that we just discovered to get to a form we could integrate. This multiplicative factor is called an **integration factor** and is generally denoted \( \mu(x) \). We still have to wonder how we could find integration factors. After all, how did we know to multiply by the factor \( x \)?

The answer is given by the following result.

**Theorem 1**

Consider a first-order linear DE given in the standard form

\[ y' + p(x)y = q(x). \quad (3) \]

Then the solution \( y(x) \) is given by

\[ y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x) \, dx \quad (4) \]

where \( \mu(x) = e^{\int p(x) \, dx} \) is the system’s integration factor.

**Proof**

By the Fundamental Theorem of Calculus and the chain rule we have that \( \mu'(x) = p(x)\mu(x) \). If we multiply the entire expression (3) by \( \mu(x) \), we have

\[ \mu(x)y' + p(x)\mu(x)y = \mu(x)q(x). \]

The LHS can be simplified by noting that

\[ \frac{d}{dx} [\mu(x)y] = \mu(x)y' + \mu'y = \mu(x)y' + p(x)\mu(x)y. \]
It follows that we have
\[
\frac{d}{dx} [\mu(x)y] = \mu(x)q(x).
\]

We can then integrate to get
\[
\mu(x)y = \int \mu(x)q(x) \, dx
\]
and isolate \( y \) to get the general solution
\[
y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x) \, dx
= e^{-\int p(x) \, dx} \int \left( e^{\int p(x) \, dx} q(x) \right) \, dx.
\]

This result is very encouraging! So long as we can evaluate the required integrals, we can always find the solution of a first-order linear DE.

**Note:** It is strongly recommended that you remember the solution method rather than the solution formula (4) (which is cumbersome to use in practice and difficult to remember). The general method is:

1. Write the DE in the standard form (3).
2. Determine the integration factor \( \mu(x) = e^{\int p(x) \, dx} \).
3. Multiply the entire expression through by \( \mu(x) \).
4. Combine the LHS by applying the product rule in reverse.
5. Integrate with respect to \( x \) and isolate for \( y \).

**Note:** It is important to have the equation in the standard form (3) where the coefficient of \( y' \) is one. Otherwise, the given integration factor method will not work (and you will likely waste a great deal of time computing incorrect integrals!).
Example 3

Solve the following IVP:
\[
\begin{align*}
\{ & y' + \frac{1}{x} \cdot y = \frac{1}{x} \\
y(1) = 0.
\end{align*}
\]

Solution: This is already in standard form, so we are ready to determine the integrating factor. We have
\[
\mu(x) = e^{\int p(x) \, dx} = e^{\int \frac{1}{x} \, dx} = e^{\ln(x)} = x.
\]

We can ignored the normally required $|x|$ in the $\ln(x)$ term by noticing that the two absolute value cases ($x > 0$ and $x < 0$) amount to multiplying the whole differential equation by a negative, which does not change it. Multiplying the entire expression by $\mu(x) = x$ gives us
\[
xy' + y = 1
\]
which we have already seen. This was our original toy example. We already know that the general solution is
\[
y(x) = 1 + \frac{C}{x}.
\]

Substituting the intial value $y(1) = 1$ gives us
\[
y(1) = 0 = 1 + C \implies C = -1.
\]

It follows that the particular solution is
\[
y(x) = 1 - \frac{1}{x}.
\]
Example 4

Solve the following IVP:

\[ \begin{cases} \ y' + y = e^{-3x} \\ y(0) = 2. \end{cases} \]

**Solution:** This is already in standard form, so we are ready to determine the integrating factor. We have

\[ \mu(x) = e^{\int p(x) \, dx} \]
\[ = e^{\int 1 \, dx} \]
\[ = e^x. \]

Multiplying the entire expression by \( \mu(x) = e^x \) gives us

\[ e^x y' + e^x y = e^x \cdot e^{-3x} = e^{-2x}. \]

Recognizing that the left-hand side now must be the product rule form (expanded out), we have

\[ \frac{d}{dx} \left[ e^x y \right] = e^{-2x}. \]

We could jump right to this if we wanted to, but it is important to recognize the intermediate step to check that we have determined the correct integration factor. We can integrate this to get

\[ \int \frac{d}{dx} \left[ e^x y \right] \, dx = \int e^{-2x} \, dx \]
\[ \implies e^x y = -\frac{e^{-2x}}{2} + C \]
\[ \implies y(x) = -\frac{e^{-3x}}{2} + Ce^{-x}. \]

Using the initial condition \( y(0) = 2 \) gives

\[ y(0) = 2 = -\frac{1}{2} + C \quad \implies \quad C = \frac{5}{2}. \]
The particular solution is therefore
\[ y(x) = -\frac{e^{-3x}}{2} + \frac{5e^{-x}}{2}. \]

**Example 5**

Solve the following IVP:
\[
\begin{cases}
  (x + 1)y' - xy = e^x \\
  y(1) = 0.
\end{cases}
\]

**Solution:** This is not in standard form, so we need to do a little work. Dividing by \((x + 1)\) we arrive at
\[ y' - \frac{x}{x + 1}y = \frac{e^x}{x + 1}. \]

In order to determine the integrating factor, we will need to determine the integral of \(-x/(x + 1)\). Using the substitution \(u = x + 1\), we have
\[
-\int \frac{x}{x + 1} \, dx = \int \frac{1-u}{u} \, du = \int \left( \frac{1}{u} - 1 \right) \, du = \ln(u) - u = \ln(x + 1) - (x + 1).
\]

Recognizing that constants (i.e. the \(-1\)) do not matter for integrating factors, we arrive at
\[ \mu(x) = e^{\ln(x+1)-x} = (x + 1)e^{-x}. \]

Multiplying the entire expression by \(\mu(x) = (x + 1)e^{-x}\) gives us
\[ (x + 1)e^{-x}y' - xe^{-x}y = 1. \]

It follows that we have
\[
\frac{d}{dx}[(x + 1)e^{-x}y] = 1
\]
which can be checked. Integrating with respect to $x$ gives

$$(x + 1)e^{-x}y = x + C$$

so that the general solution is

$$y(x) = \frac{e^x}{x + 1} (x + C).$$

The initial condition $y(1) = 0$ gives

$$y(1) = 0 = \frac{e}{2}(1 + C) \implies C = -1.$$ 

It follows that the particular solution is

$$y(x) = e^x \left( \frac{x - 1}{x + 1} \right).$$

Section 3: Substitution Methods

Most first-order differential equations, even those with simple solutions, do not fall directly within the two classes we have studied so far (separable and first-order linear). Consider the following example.

**Example 6**

Find the general solution of

$$y' = \frac{x^2 + y^2}{2xy}.$$ 

**Solution:** This differential equation is neither separable nor first-order linear (check!). In order to solve it, we will take a page from our Calculus courses. Recall that very few integrals naturally appeared in a form where they could be integrated directly, rather, we had to first perform a variable substitution.

The study of differential equations is no different! What we will do is
introduce a new variable \( v(x, y) \) and rewrite the DE in \( x \) and \( y \) as a DE in terms of \( v \) and \( x \). For this example, consider the variable substitution

\[
v(x, y) = \frac{y}{x}.
\]

(We do not need to know why we chose this substitution—yet.) We can now transform the differential equation by using

\[
y = xv \implies \frac{dy}{dx} = x \frac{dv}{dx} + v.
\]

The differential equation can be rewritten in terms of \( v \) and \( x \) as

\[
2x(xv) \left( x \frac{dv}{dx} + v \right) = x^2 + (xv)^2
\]

\[
\implies 2x^3 v \frac{dv}{dx} = x^2 + x^2 v^2 - 2x^2 v^2
\]

\[
\implies 2x^3 v \frac{dv}{dx} = x^2 (1 - v^2)
\]

\[
\implies \frac{2v}{1 - v^2} dv = \frac{1}{x} dx.
\]

After looking at this equation for a moment, we realize something amazing has happened—this DE is separable even though the original DE was not. We can integrate to obtain

\[
\int \frac{2v}{1 - v^2} dv = \int \frac{1}{x} dx
\]

\[
\implies - \ln(1 - v^2) = \ln(x) + C,
\]

\[
\implies 1 - v^2 = \frac{\tilde{C}}{x}.
\]

We can now to return to the original variables \( x \) and \( y \). We started with \( v = y/x \), so we now have

\[
1 - \left( \frac{y}{x} \right)^2 = \frac{\tilde{C}}{x}
\]
This yields the final solution
\[ y = \pm \sqrt{x^2 - \tilde{C}x}. \]

This was a little work, but the end result is very satisfying. The real question is how we knew to use the substitution \( v = y/x \). In fact, this DE belongs to a general class of systems known as **(power) homogeneous equations** for which this substitution will *always* yield a separable equation in the variables \( v \) and \( x \). We will also consider a general class of systems known as **Bernoulli equations** for which an appropriate variable substitution always yields a first-order linear DE.

**Note:** It is hoped that it is the *method* which is memorized, not the end formula. In other words, remember the required variable substitutions for the types of equations rather than the final expression for the solution.

**Note:** As with any problem involving layers of variable dependences, it is helpful to write out the tree of variable dependences. For the differential equations we are looking at, where we are looking for a function \( y = y(x) \) (i.e. \( y \) as a function of \( x \)) and using a variable transformation \( v = v(x, y) \), we have the tree

\[
\begin{align*}
\text{v} & \quad \rightarrow \quad \text{y} \\
\text{x} & \quad \rightarrow \quad \text{y} \\
\text{x} & \quad \rightarrow \quad \text{x}
\end{align*}
\]

which gives the following derivative (by the multivariate chain rule):
\[
\frac{dv}{dx} = \frac{\partial v}{\partial y} \frac{dy}{dx} + \frac{\partial v}{\partial x}. \]
We start with the following definition.

**Definition 2**
A first-order differential equation is called *(power)* homogeneous if it has the form

\[ \frac{dy}{dx} = F\left(\frac{y}{x}\right). \]

A substitution of the form \( v = \frac{y}{x} \) produces a separable differential equation in \( v \) and \( x \) of the form

\[ x \frac{dv}{dx} = F(v) - v. \]

**Note:** *(Power)* homogeneous equations are more easily recognized by the observation that the combined powers of \( x \) and \( y \) in each term (i.e. the sum of the powers) must be the same for all terms. For instance

\[ y' = \frac{x^{1/2}y^{3/2} + y^2 + xy}{x^{3/2}y^{1/2} + x^2} \]

is *(power)* homogeneous because every term has the effective power of 2. Note that this must hold for both the numerator and the denominator in the RHS. For example, \( y' = x^2 + y^2 \) is *not* *(power)* homogeneous because the denominator (not shown) has the effective power of zero.

**Note:** Within the study of differential equations there are *two* accepted definitions of what constitutes a homogeneous differential equation, and these definitions are very different. Usually context will dictate which meaning is implied, but just to be clear we will use *(power)* homogeneous to refer to the class of differential equations we just introduced.
Example 7

Determine the general solution of \( y' = \frac{y + 2\sqrt{xy} + x^2}{x} \).

**Solution:** We can quickly check that this DE is neither separable nor first-order linear. We can, however, observe that the effective power of each term is one. This is obvious for the terms \( y \) and \( x \), but it also holds for the terms \( xy \) and \( x^2 \) under the square root. If we prefer, we can divide \( x \) through the RHS explicit to obtain

\[
y' = \frac{y}{x} + 2\sqrt{\frac{y}{x} + 1} = F\left(\frac{y}{x}\right)
\]

where \( F(v) = v + 2\sqrt{v + 1} \).

We may now solve the DE with the substitution \( v = y/x \). This gives \( y = xv \) so that \( y' = v + xv' \). It follows that we have

\[
v + xv' = v + 2\sqrt{v + 1}
\implies xv' = 2\sqrt{v + 1}
\implies \int \frac{1}{2\sqrt{v + 1}} dv = \int \frac{1}{x} dx
\implies \sqrt{v + 1} = \ln|x| + C.
\]

We now return to our original variables \( y \) and \( x \). We have

\[
\sqrt{\frac{y}{x} + 1} = \ln|x| + C
\]

so that

\[
y(x) = x \left[ (\ln|x| + C)^2 - 1 \right].
\]

Section 5: Bernoulli DEs

We now consider the following class of systems.
Definition 3
A first-order differential equation is Bernoulli if it has the form

\[ y' + P(x)y = Q(x)y^n \]

for \( n \neq 0, 1 \). A substitution of the form \( v = y^{1-n} \) produces a first-order linear differential equation in \( v \) and \( x \) of the form

\[ v' + (1 - n)P(x)v = (1 - n)Q(x). \]

Bernoulli equations are close to first-order linear in the sense that, if we rearrange the equation to the standard form for first-linear linear DEs, the only difference is a single additional term which is a power of \( y \). Rearranging to standard form will be our typical method for determining whether a DE in of Bernoulli form.

**Note:** It is worth noting that the substitution \( v = y^{1-n} \) holds for all values of \( n \) other than \( n = 0 \) and \( n = 1 \). That is to say, we can consider fractional powers (e.g. \( n = 1/2, n = 7/5, n = 92/13, \) etc.) and negative powers (\( n = -3, n = -4/9, n = -103, \) etc.). The values \( n = 0 \) and \( n = 1 \) are degenerate for the substitution, but correspond to trivial cases for the original DE since it is already first-order linear.

Example 8
Determine the solution of

\[ 3xy^2y' = 3x^4 + y^3. \]

**Solution:** It should again not take much argument to convince ourselves that this equation is not separable, is not first-order linear, and is not even power homogeneous (although it is close). To check whether it is Bernoulli, we rearrange this equation to get it as close to the first-order linear form as possible. We have

\[ y' = x^3y^{-2} + \frac{1}{3x}y \implies y' - \frac{1}{3x}y = x^3y^{-2}. \]
The only term which distinguishes this from first-order linear is the one on the right-hand side. We are not happy with the $y^{-2}$ and would like to make it go away.

We use the substitution $v = y^{1-n} = y^3$ where $n = -2$. We want to rewrite this differential equation in $y$ and $x$ as a differential equation in $v$ and $x$. We have

$$y = v^{1/3} \implies \frac{dy}{dx} = \left(\frac{1}{3}v^{-2/3}\right)\frac{dv}{dx}$$

and $y^{-2} = v^{-2/3}$. It follows that the differential equation can be rewritten as

$$\left(\frac{1}{3}v^{-2/3}\right)v' - \frac{1}{3x}v^{1/3} = x^3v^{-2/3}.$$

Multiplying across by $3v^{2/3}$ we arrive at

$$v' - \frac{1}{x}v = 3x^3.$$

Remarkably, the trouble term $y^{-2}$ has disappeared. In fact, this is now a linear equation in $v$ and $x$ which we can solve. We have the integrating factor

$$\mu(x) = e^{-\int \frac{1}{x} \, dx} = e^{-\ln(x)} = \frac{1}{x}.$$ 

This gives us

$$\frac{1}{x}v' - \frac{1}{x^2}v = 3x^2 \implies \frac{d}{dx}\left[\frac{1}{x}v\right] = 3x^2$$

$$\implies \quad \frac{1}{x}v = \int 3x^2 \, dx = x^3 + C \quad \implies \quad v = x^4 + Cx.$$

We are not completely done. The original DE was with respect to $y$ and $x$, so we need to change our solution back to these original variables. We have

$$y^3 = x^4 + Cx \implies y(x) = \sqrt[3]{x^4 + Cx}.$$
Suggested Problems:

1. Solve the following first-order IVPs: (If it is possible to solve as both separable and first-order linear, consider solving by both methods!)
   
   (a) \[
   \begin{cases}
   y' = y^2 - 5y + 4 \\
   y(0) = 1
   \end{cases}
   \]
   
   (b) \[
   \begin{cases}
   y' = x(y - 1) \\
   y(1) = 2
   \end{cases}
   \]
   
   (c) \[
   \begin{cases}
   y' = e^{x+y} \\
   y(0) = 0
   \end{cases}
   \]
   
   (d) \[
   \begin{cases}
   y' = ay + b, \ a, b \in \mathbb{R} \\
   y(x_0) = y_0, \ x_0, y_0 \in \mathbb{R}
   \end{cases}
   \]

   (e) \[
   \begin{cases}
   y' = -\sin(x)y + \sin(2x) \\
   y(\pi) = 0
   \end{cases}
   \]
   
   (f) \[
   \begin{cases}
   y' = \frac{y+x}{y+1} \\
   y(0) = 1
   \end{cases}
   \]
   
   (g) \[
   \begin{cases}
   y' = \frac{x+1}{y+1} \\
   y(0) = -1
   \end{cases}
   \]
   
   (h) \[
   \begin{cases}
   y' = -2\frac{y}{x} + \ln(x) \\
   y(0) = 0
   \end{cases}
   \]

2. Use a substitution method to find the general solution of the following DEs:

   (a) \[
   y' = \frac{5y^4 - 2xy}{x^2}
   \]
   
   (b) \[
   y' = \frac{y + x}{y - x}
   \]
   
   (c) \[
   y' = \frac{y^2}{x^2 + 2xy}
   \]
   
   (d) \[
   y' = \frac{3xy^{4/3} - 6y}{x}
   \]
   
   (e) \[
   y' = \frac{4xy^{4/3} + y}{x}
   \]
   
   (f) \[
   y' = -\frac{y + x^2y^2e^{2x}}{x}
   \]
   
   (g) \[
   y' = \frac{y + 2\sqrt{xy}}{x}
   \]
   
   (h) \[
   y' = \frac{2y^2 + x\sqrt{x^2 + y^2}}{2xy}
   \]