One application of first-order linear differential equations differential equations is modeling the amount (or concentration) of a substance in a well-stirred tank/vessel subject to constant in-flow and out-flow. Common simple applications are:

1. an industrial mixing tank with an entry pipe and an exit pipe;

2. a lake with an inflowing river feeding a pollutant in from upstream and an outflowing river removing pollutant downstream;

3. a tub or sink with a flowing faucet and open drain.

In all cases, we are interested in the modeling the amount of substance in the mixing vessel over time.

We want to translate this description into mathematics. The fundamental equation we will use will be

\[ \text{[rate of change]} = \text{[rate in]} - \text{[rate out]} \]

That is to say, at each instance in time, we believe that the rate of change of the overall amount of the quantity of interest to equal the amount that is flowing in minus the amount that is flowing out.
To characterize the inflow rate, we need to know the overall flow rate and concentration of the substance of interest in inflowing mixture. This could be the amount of pollutant in an inflowing stream, or the amount of chemical diluted in an inflowing pipe. We have

$$[\text{rate in}] = [\text{concentration}] \cdot [\text{volume rate in}]$$

since

$$[\text{concentration}] \cdot [\text{volume rate in}] = \left[\frac{\text{amount}}{\text{volume}}\right] \cdot \left[\frac{\text{volume}}{\text{time}}\right] = \left[\frac{\text{amount}}{\text{time}}\right] \quad (1)$$

The outflow is slightly different. We again need to know the flow rate and concentration of substance in the outflowing mixture. We do not, however, know the amount or concentration of the substance in the system—that is what we are trying to find! We will have to make some assumptions. The most basic will be to assume that the vessel is well-mixed so that that the concentration of the substance is uniform throughout the mixing vessel. That is, we do not allow pockets of high concentration of pollutants or pockets of low concentration.

Give this, by (1) we have

$$[\text{rate out}] = [\text{concentration}] \cdot [\text{volume rate out}] = \left[\frac{\text{amount}}{\text{volume}}\right] \cdot [\text{volume rate out}].$$

Letting $A$ denote the amount of substance in the tank and $V(t)$ denote the current volume, we have the combined model

$$\frac{dA}{dt} = [\text{volume rate in}] \cdot [\text{concentration in}] - \frac{A}{V(t)} \cdot [\text{volume rate out}].$$

**Note:** The key difference between the inflow and outflow rates is that the amount and volume in the outflow rate depend upon the current amount and volume in the mixing vessel. In the inflow tank, these quantities are either controlled (for mixing tanks) or known (for rivers and streams). Notice also that if the volume of the in-flow and the volume of the out-flow do not balance, the volume of the tank may be a dynamic function of time (imagine filling a bathtub, or emptying a mixing tank).
Example 1
Suppose that there is a factory built upstream of a lake with a volume of 0.5 km$^3$. The factory introduces a new pollutant to a stream which pumps 1 km$^3$ of water into the lake every year. Suppose that the net outflow from the lake is also 1 km$^3$ per year and that the concentration of the pollutant in the inflow stream is 200 kg/km$^3$.

(a) Set up an initial value problem for the amount of pollutant in the lake and solve it.

(b) Assuming there is initially no pollutant in the lake, how much pollutant is there are one month?

(c) What is the limiting pollutant level?

Solution: We need to set up the model in the form $\text{[rate of change]} = \text{[rate in]} - \text{[rate out]}$. If we let $A$ denote the amount of the pollutant (in kg), we have

$$dA \over dt.$$  

In order to determine the rate in, we notice that the amount (in kg) coming from the inflow can be given by

$$\text{rate in} = \text{concentration in} \cdot \text{volume rate in}$$

$$= (200 \text{ kg/km}^3) \cdot (1 \text{ km}^3/\text{year}) = 200 \text{ kg/year}.$$  

The rate out is given by

$$\text{rate out} = \text{concentration out} \cdot \text{volume rate out}$$

$$= \left( {A \over 0.5} \text{ kg/km}^3 \right) (1 \text{ km}^3/\text{year}) = 2A \text{ kg/year}.$$  

We can see the units have worked as desired. We can drop them and just focus on the initial value problem

$$dA \over dt = 200 - 2A, \quad A(0) = A_0.$$  

3
This is a first-order linear differential equation which in standard form is given by

\[ \frac{dA}{dt} + 2A = 200. \]

We can see that we have \( p(x) = 2 \) and \( q(x) = 200 \). The necessary integration factor is

\[ \mu(t) = e^{\int 2 \, dt} = e^{2t} \]

so that we have

\[ e^{2t} \frac{dA}{dt} + 2e^{2t}A = 200e^{2t} \]

\[ \implies \frac{d}{dt} [e^{2t} A] = 200e^{2t} \]

\[ \implies e^{2t} A = \int 200e^{2t} \, dt = 100e^{2t} + C \]

\[ \implies A(t) = 100 + Ce^{-2t}. \]

In order to solve for \( C \), we use \( A(0) = A_0 \) to get

\[ A(0) = A_0 = 100 + C \implies C = A_0 - 100. \]

This gives the solution

\[ A(t) = 100 + (A_0 - 100)e^{-2t}. \]

For this form, we can easily answer part (b). Given an initial pollutant level of zero (i.e. \( A_0 = 0 \)), we have

\[ A(t) = 100 - 100e^{-2t}. \]

After one month has passed, we have \( t = 1/12 \) so that the amount of pollutant is given by

\[ A(1/12) = 100 - 100e^{-2(1/12)} \approx 15.3528 \text{ kg}. \]

We can also easily determine the limiting pollutant level by evaluating

\[ \lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left[ 100 + (A_0 - 100)e^{-2t} \right] = 100. \]
In other words, no matter what the initial amount is in the lake, we will always converge toward 100 kg of pollutant distributed throughout the lake. This makes sense. The limiting level is going to be when the rate in and the rate out are balanced. That occurs for this model when \(200 = 2A\) which implies \(A = 100\).

**Example 2**

Consider a 50 gallon tank which is initially filled with 20 gallons of brine (salt/water mixture) with a concentration of \(1/4\) lbs/gallon of salt. Suppose that there is an inflow tube which infuses 3 gallons of brine into the tank per minute with a concentration of 1 lbs/gallon. Suppose that there is an outflow tube which flows at a rate of 2 gallons per minute.

(a) Set up and solve a differential equation for the amount of salt in the tank.

(b) How much salt is in the tank when the tank is full?

**Solution:** This is slightly different than the previous example because the volume of mixture in the tank changes because the inflow and outflow volume rates are different. There is more mixture flowing into the tank than flowing out. Nevertheless, we can incorporate this into our model by noting that the volume of the tank at time \(t\) can be given by

\[V(t) = 20 + (3 - 2)t = 20 + t.\]

We can now complete the model as before. We have

\[
\frac{dA}{dt} = (3)(1) - (2) \frac{A}{20 + t} = 3 - \frac{2A}{20 + t}, \quad A(0) = 20(1/4) = 5.
\]

Again, this is a first-order linear differential equation. We can solve it by rewriting

\[
\frac{dA}{dt} + \left(\frac{2}{20 + t}\right) A = 3
\]

and determining the integrating factor

\[
\mu(t) = e^{\int \frac{2}{20+t} \, dt} = e^{2 \ln(20+t)} = (20 + t)^2.
\]
This gives
\[
(20 + t)^2 \frac{dA}{dt} + 2(20 + t)A = 3(20 + t)^2
\]
\[
\implies \frac{d}{dt} [(20 + t)^2 A] = 3(20 + t)^2
\]
\[
\implies (20 + t)^2 A = (20 + t)^3 + C
\]
\[
\implies A(t) = (20 + t) + \frac{C}{(20 + t)^2}.
\]
Using the initial condition \(A(0) = 5\), we have
\[
A(0) = 5 = 20 + \frac{C}{400} \implies C = -6000
\]
so that the particular solution is
\[
A(t) = (20 + t) - \frac{6000}{(20 + t)^2}.
\]
To answer the question of how much salt will be in the tank when the tank is full, we notice that the tank will be full when \(V(t) = 20 + t = 50\), which implies \(t = 30\) (i.e. it will take thirty minutes). This gives
\[
A(30) = (20 + 30) - \frac{6000}{(20 + 30)^2} = 50 - \frac{6000}{2500} = 47.6.
\]
It follows that there will be 47.6 lbs of salt in the tank when it is full.

Section 2: Second-Order Systems

We now move on to consideration of the following class of differential equations.

Definition 1

A second-order linear differential equation is given by the form
\[
y'' + p(x)y' + q(x)y = g(x)
\]
(2)
where \( y = y(x) \). The equation (6) is furthermore said to be:

1. **constant coefficient** if \( p(x) \) and \( q(x) \) are constants:
   \[
   ay'' + by' + cy = g(x),
   \] (3)

2. **homogeneous** if \( g(x) = 0 \):
   \[
   y'' + p(x)y' + q(x)y = 0.
   \] (4)

We can think of second-order linear differential equations as an extension of first-order DEs of the form

\[
y' + p(x)y = q(x).
\] (5)

In this case, however, we allow the second-order derivative \( y'' \) to appear in the expression. While this may seem like a small generalization, in fact, it will render most of the methods relevant to analyzing the first-order DE (5) moot. In particular, the integrating factor solution method will not apply, and we will not be able to build a slope field diagram to visualize solutions prior to consideration of the solutions. Second-order differential equations are nevertheless among the most commonly encountered form of differential equations in the sciences. For example, they arise directly as a consequence of Newton’s second law of motion \( F = ma \) in classical physics.

We start our consideration of second-order differential equations with the easiest possible case: **linear, homogeneous, and constant coefficient**. Such a differential equation can be written in the general form

\[
ay'' + by' + cy = 0.
\] (6)

We already know that such a differential equation can be rewritten as a linear system in two variables and then solved, but suppose we did not know that. We cannot separate the variables, or find an integrating factor, or find an obvious substitution which will reduce the differential equation to first-order. This exhausts our list of solution methods.

In fact, we will simply take an **educated guess**. We know that the first-order system

\[
y' = ry
\]

has the exponential solution \( y(x) = e^{rx} \), so we will guess that a solution of (6) has the form

\[
y(x) = e^{rx}
\]
for some \( r \in \mathbb{R} \). In the worst case scenario, even if this does not work out, we have not lost a great deal of time—taking derivatives of exponentials is easy.

Consider the following example.

### Example 3

Find a solution of the following second-order differential equation in the form \( y(x) = e^{rx} \):

\[
y'' - 5y' + 4y = 0. \tag{7}
\]

**Solution:** We will guess that the solution has the form \( y(x) = e^{rx} \). This gives

\[
y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}
\]

so that

\[
y'' - 5y' + 4y = r^2 e^{rx} - 5re^{rx} + 4e^{rx} = e^{rx}(r^2 - 5r + 4) = e^{rx}(r - 1)(r - 4) = 0.
\]

The exponential is always positive, so we have either \( r = 1 \) or \( r = 4 \). It follows that

\[
y_1(x) = e^x \quad \text{and} \quad y_2(x) = e^{4x}
\]

are solutions of the differential equation.

In fact, we can easily verify this. For \( y_1(x) = e^x \), we have

\[
y_1'' - 5y_1' + 4y = e^x - 5e^x + 4e^x = 0
\]

and for \( y_2(x) = e^{4x} \), we have

\[
y_2'' - 5y_2' + 4y = 16e^{4x} - 5(4e^{4x}) + 4e^{4x} = 0.
\]

We might be initially surprised that we have two solutions to the equation, but we should not be. Since we have two initial conditions for second-order equations, we will also have two independent solutions, and each one will come with an undetermined constant.

In fact, this is a general property which applies to all homogeneous linear systems. We have the following result, which is known as the **Principle of**
Superposition.

**Theorem 1**
Suppose that \( y_1(x) \) and \( y_2(x) \) are solutions of
\[
y'' + p(x)y' + q(x)y = 0. \tag{8}
\]
Then \( y = C_1y_1 + C_2y_2 \) where \( C_1, C_2 \in \mathbb{R} \) is a solution of (8).

**Proof**
Since \( y_1(x) \) and \( y_2(x) \) are solutions of (8), it follows that
\[
y_1'' + p(x)y_1' + q(x)y_1 = 0 \quad \text{and} \quad y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{9}
\]
We will now check if \( y(x) = C_1y_1(x) + C_2y_2(x) \) is a solution of (8). Note first of all that we have
\[
y'(x) = C_1y_1'(x) + C_2y_2'(x) \quad \text{and} \quad y''(x) = C_1y_1''(x) + C_2y_2''(x).
\]
On the left-hand side of (8), we therefore have
\[
y'' + p(x)y' + q(x)y = [C_1y_1'' + C_2y_2''] + p(x) [C_1y_1' + C_2y_2'] + q(x) [C_1y_1 + C_2y_2]
= C_1 [y_1'' + p(x)y_1' + q(x)y_1] + C_2 [y_2'' + p(x)y_2' + q(x)y_2]
= 0
\]
where the last line follows from (9). Connecting the first line to the final one, we have that \( y'' + p(x)y' + q(x)y = 0 \) so that \( y(x) \) is a solution of (8), and we are done.

**Note:** The principle of superposition does not require the equation to have **constant coefficients**. It is easy, however, to find violations of Theorem 1 for equations which are not **linear** or not **homogeneous**. For example, the nonlinear (but homogeneous) DE \( y' - 2y^{1/2} = 0 \) has the solution \( y(x) = x^2 \) but \( \tilde{y}(x) = Cx^2 \) is not a solution for \( C \neq \pm 1 \).
Similarly, the nonhomogeneous (but linear) DE \( y' - y = e^x \) has the solution \( y(x) = xe^x \) but \( \tilde{y}(x) = Cxe^x \) is not a solution for \( C \neq 1 \).

The principle of superposition clarifies our earlier concern about solutions to the second-order equation (6). We are not completely done, however. Consider guessing the solution form \( y(x) = e^{rx} \) for the general form (6). We have

\[
ay'' + by' + cy = 0 \implies e^{rx}(ar^2 + br + c) = 0 \implies ar^2 + br + c = 0.
\]

If we cannot factor this expression, we need to use the quadratic formula. This gives

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

If there are two real-valued roots, we obtain the two fundamental solutions \( y_1(x) = e^{r_1x} \) and \( y_2(x) = e^{r_2x} \). If the roots are complex or repeated, however, we will need alternative solution forms. We have the following result.

**Theorem 2**

The general solution of the second-order DE (6) is given by the following:

1. If \( b^2 - 4ac > 0 \) the general solution is \( y(x) = C_1 e^{r_1x} + C_2 e^{r_2x} \) where \( r_1 \) and \( r_2 \) are the two distinct roots of (10).

2. If \( b^2 - 4ac = 0 \) the general solution is \( y(x) = C_1 e^{rx} + C_2 xe^{rx} \) where \( r \) is the single repeated root of (10).

3. If \( b^2 - 4ac < 0 \) the general solution is

\[
y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))
\]

where \( r = \alpha + \beta i \).

Consider the following examples.

**Example 4**

Find the general solution of \( 4y'' + 12y' + 9y = 0 \). Then find the particular solution for \( y(0) = 2 \) and \( y'(0) = 0 \).
Solution: We guess the solution form $y(x) = e^{rx}$. This gives

$$4y'' + 12y' + 9y = e^{rx}(4r^2 + 12r + 9) = e^{rx}(2r + 3)^2 = 0.$$ 

It follows that we only have a solution if $r = -3/2$. Since this is a repeated root, we are in Case 2 and the general solution is given by

$$y(x) = C_1e^{-(3/2)x} + C_2xe^{-(3/2)x}.$$

To solve for the particular solution, we compute

$$y'(x) = -\frac{3}{2}C_1e^{-(3/2)x} + C_2e^{-(3/2)x} - \frac{3}{2}C_2xe^{-(3/2)x}.$$

The conditions $y(0) = 3$ and $y'(0) = 0$ give the system

$$\begin{align*}
C_1 &= 2 \\
-\frac{3}{2}C_1 + C_2 &= 0.
\end{align*}$$

We can quickly solve this to get $C_1 = 2$ and $C_2 = 3$. It follows that the particular solution is $y(x) = 2e^{-(3/2)x} + 3xe^{-(3/2)x}$.

Example 5

Find the general solution of $y'' - 2y' + 5y = 0$. Then find the particular solution for $y(0) = 1$ and $y'(0) = -1$. 

11
Solution: We guess the solution $y(x) = e^{rx}$. This gives

$$y'' - 2y' + 5y = e^{rx}(r^2 - 2r + 5) = 0.$$ 

The quadratic formula gives the solution

$$r = \frac{2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$ 

Since this a complex root, we are in case 3 with $\alpha = 1$ and $\beta = 2$. The general solution is

$$y(x) = C_1 e^x \cos(2x) + C_2 e^x \sin(2x).$$ 

To solve for the particular solution, we compute

$$y'(x) = C_1 e^x \cos(2x) - 2C_2 e^x \sin(2x) + C_1 e^x \sin(2x) + 2C_2 e^x \cos(2x)$$
$$= C_1 e^x (\cos(2x) + \sin(2x)) + 2C_2 e^x (\cos(2x) - \sin(2x)).$$ 

The conditions $y(0) = 1$ and $y(0) = -1$ give the system

$$C_1 = 1$$
$$C_1 + 2C_2 = -1.$$ 

It follows immediately that $C_1 = 1$ and $C_2 = -1$ so that the particular solution is $y(x) = e^x(\cos(2x) - \sin(2x))$.
We can observe the components the solution in the features of the graph. The solution grows because of the exponential term $e^x$, and oscillates because of the trigonometric terms $\sin(2x)$ and $\cos(2x)$. Terms like these, and the associated qualitative behaviors, will be key features of solutions of this type of differential equation.

## Suggested Problems

1. Consider a mixing tank with a total volume of 20 gallons, initially filled with 10 gallons of pure water. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water) mixture at a rate of 4 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallon per minute.

   (a) Use the given information to derive a differential equation which models the amount of salt in the tank.

   (b) Find the general solution of the differential equation derived in part (a).

   (c) How much salt is in the tank when it is full?

2. Consider a filled mixing tank with a volume of 20 gallons. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water mixture) at a rate of 2 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallons per minute.

   (a) Use the given information to derive a differential equation which models the amount of salt in the tank.

   (b) Find the general solution of the differential equation derived in part (a).

   (c) Suppose there is initially no salt in the tank. How much salt is in the tank after ten minutes?

3. Suppose a factory is built upstream of a lake with a volume of 0.5 km$^3$. The factory introduces a pollutant into the upstream water system. Suppose the affected water system pumps 0.25 km$^3$ of water into the
lake each year and the downstream water system removes water from the lake at the same rate. Suppose the concentration of pollutant in the feeding water system is 40 kg/km$^3$.

(a) Set up a first-order differential equation which models the amount of pollutant (in kg) in the lake.

(b) Suppose that there is initially no pollutant in the lake. How much pollutant is in the lake after (i) one month; (ii) seven months; (iii) five years? What is the limiting amount of pollutant in the lake?

(c) Suppose now that the inflow and outflow rates of the upstream and downstream water systems vary based on the seasons. Suppose this variance can be modeled by the form [volume rate in] = [volume rate out] = 0.25 + 0.25\cos(2\pi t). Derive the corresponding first-order differential equation which models the amount of pollutant (in kg) in Lake Mendota.

(d) Suppose that there is initially no pollutant in the lake. Under the assumptions of part (c), determine how much pollutant is in the lake after (i) one month; (ii) seven months; (iii) five years. What is the limiting amount of pollutant in the lake? Is it the same as in part (b)? Does it converge to this value faster or slower than part (b)? Provide a brief explanation for the observed differences.

4. Solve the following second-order linear differential equations (all derivatives with respect to $x$):

(a) \[ \begin{cases} y'' - 7y' + 10y = 0, \\ y(0) = 3, \\ y'(0) = 0 \end{cases} \]

(b) \[ \begin{cases} 4y'' - 12y' + 9y = 0, \\ y(0) = 1, \\ y'(0) = -1/2 \end{cases} \]

(c) \[ \begin{cases} y'' + 4y' + 5y = 0, \\ y(0) = 1, \\ y'(0) = 1 \end{cases} \]

(d) \[ \begin{cases} 2y'' - 9y' + 7y = 0, \\ y(0) = 0, \\ y'(0) = 5 \end{cases} \]

(e) \[ \begin{cases} 4y'' + 4y' + 5y = 0, \\ y(0) = -1, \\ y'(0) = \frac{3}{2} \end{cases} \]

(f) \[ \begin{cases} y'' - 4y' + 4y = 0, \\ y(0) = -1, \\ y'(0) = 1 \end{cases} \]

(g) \[ \begin{cases} 4y'' - 3y' - y = 0, \\ y(0) = 1, \\ y'(0) = -\frac{3}{2} \end{cases} \]

(h) \[ \begin{cases} y'' + 4y' + 29y = 0, \\ y(0) = 2, \\ y'(0) = -2 \end{cases} \]
5. Differential equations of the form

\[ ax^2 y'' + bxy' + cy = 0 \]

can be transformed into a constant coefficient system by the variable transformation \( v = \ln(x) \). Use this transformation to solve the following non-constant coefficient second-order systems.

\[
\begin{align*}
\text{(a)} \quad & \begin{cases} 
    x^2 y'' + 9xy' + 15y = 0, \\
    y(1) = 1, \\
    y'(1) = -3
\end{cases} & \begin{cases} 
    x^2 y'' + xy' + 25y = 0, \\
    y(1) = 1, \\
    y'(1) = 5
\end{cases}
\end{align*}
\]