Determining the inverse Laplace transform of $F(s)$ will commonly involve partial fraction decomposition. We quickly review the steps for attaining the correct form for expansion.

**Algorithm 1**

Consider a function $F(s) = \frac{f(s)}{g(s)}$ where $f$ and $g$ are polynomials, i.e. $f(s) = a_0 + a_1s + \cdots + a_ms^m$ and $g(s) = b_0 + b_1s + \cdots + b_ns^n$ ($m < n$).

The appropriate steps for obtaining the partial fraction decomposition are the following:

1. Fully factor $g(s)$. The Fundamental Theorem of Algebra guarantees that any polynomial can be factored uniquely into chains of terms of one of two forms:

   $$(as + b)^n \quad \text{or} \quad (as^2 + bs + c)^n.$$  

   Every polynomial with a power of 3 of higher can be factored!

2. For every term of the form $(as + b)^n$ in $g(s)$, add the following chain of terms

   $$\frac{A_1}{as + b} + \frac{A_2}{(as + b)^2} + \cdots + \frac{A_n}{(as + b)^n} \quad (1)$$

   on the right-hand side. For every term of the form $(as^2 + bs + c)^n$, add the following chain

   $$\frac{B_1s + C_1}{as^2 + bs + c} + \frac{B_2s + C_2}{(as^2 + bs + c)^2} + \cdots + \frac{B_ns + C_n}{(as^2 + bs + c)^n} \quad (2)$$

Note that if $n = 1$, we only add a single term.
Example 1

Set up the partial fraction expansion of
\[
\frac{1}{(s-1)(2s+3)^3(s^2+s+1)^2(3s^2-s+3)}.
\]
Do not evaluate for the constants!

Solution: This is a straight forward application of (1) and (2). We have
\[
\frac{1}{(s-1)(2s+3)^3(s^2+s+1)^2(3s^2-s+3)} = \frac{A}{s-1} + \frac{B}{2s+3} + \frac{C}{(2s+3)^2} + \frac{D}{(2s+3)^3} + \frac{Es+F}{s^2+s+1} + \frac{Gs+H}{(s^2+s+1)^2} + \frac{Is+J}{3s^2-s+3}.
\]

Example 2

Perform partial fraction decomposition on the following:
\[
F(x) = \frac{s^2+1}{s^3-1}.
\]

Solution: Since \(g(s) = s^3-1\) is third order, it can be factored. In this case, we have the obvious root \(s = 1\) so that \((s-1)\) is a factor. We could also apply difference of cubes. We have
\[
\frac{s^2+1}{s^3-1} = \frac{s^2+1}{(s-1)(s^2+s+1)}.
\]
We set-up our partial fraction decomposition for three variables \(A\), \(B\), and \(C\) so that
\[
\frac{s^2+1}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}.
\]
Multiplying across by the denominator on the left-hand side we arrive at the more manageable form

\[ s^2 + 1 = A(s^2 + s + 1) + (Bs + C)(s - 1). \]

In order for our partial fraction decomposition to be valid, the above expression must hold for all values of \( s \). This suggests two alternative methods of solving for the constants.

1. Plug values of \( s \) into the equation until you have enough expressions to solve for the variables. Particularly useful values of \( s \) are those which eliminate brackets (e.g. if \((s - 2)\) appears factored several times, select \( s = 2 \)).

2. Collect powers of \( s \) on the right-hand side and then equate coefficients on the left-hand and right-hand side.

For illustrative purposes, we will perform both methods here. To the first method, we select the values \( s = 0 \), \( s = 1 \) and \( s = -1 \). Plugging \( s = 0 \) into the expression, we have \( 1 = A - C \) which implies \( C = A - 1 \). Plugging in \( s = 1 \) gives \( 2 = 3A \) which implies \( A = \frac{2}{3} \), and therefore that \( C = -\frac{1}{3} \). Plugging in \( s = -1 \) gives \( 2 = A - 2C + 2B \). We can plug in our known values of \( A \) and \( C \) and solve for \( B \) to get \( B = \frac{1}{3} \).

Alternatively, we can expand our original expression to get

\[ s^2 + 1 = (A + B)s^2 + (A - B + C)s + (A - C). \]

Equating the coefficients of the \( s \) terms on the left-hand and right-hand side (realizing that \( s^2 + 1 = (1)s^2 + (0)s + (1) \)) gives the system of equations

\[
\begin{align*}
A + B &= 1 \\
A - B + C &= 0 \\
A - C &= 1.
\end{align*}
\]

For those who are familiar with matrix analysis, this can be solved through row reduction. Otherwise, we back substitute variables to get \( C = A - 1 \) \(\Rightarrow\) \( A - B + C = 2A - B = 1 \) \(\Rightarrow\) \( B = 2A - 1 \) \(\Rightarrow\) \( A + B = 3A = 2 \) \(\Rightarrow\) \( A = \frac{2}{3} \) \(\Rightarrow\) \( B = \frac{1}{3} \) \(\Rightarrow\) \( C = -\frac{1}{3} \).
We put this together to get

$$\frac{s^2 + 1}{(s - 1)(s^2 + s + 1)} = \frac{2}{3(s - 1)} + \frac{s - 1}{3(s^2 + s + 1)}.$$